Reconfiguration of Hamiltonian Cycles in Rectangular Grid Graphs

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January 11, 2026

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Abstract

An $m \times n$ *grid graph* is the induced subgraph of the square lattice whose vertex set consists of all integer grid points $\{(i, j) : 0 \le i < m,\ 0 \le j < n\}$. Let H and K be Hamiltonian cycles in an $m \times n$ grid graph G. We study the problem of reconfiguring H into K, • where the Hamiltonian cycles are viewed as vertices of a reconfiguration graph •, using a sequence of local transformations called *moves*. A *box* of G is a unit square face. A box with vertices a, b, c, d is *switchable* in H if exactly two of its edges belong to H, and these edges are parallel. Given such a box with edges ab and cd in H, a *switch move* removes ab and cd, and adds bc and ad. A *double-switch move* consists of performing two consecutive switch moves. If, after a double-switch move, we obtain a Hamiltonian cycle, we say that the double-switch move is *valid*.

We prove that any Hamiltonian cycle H can be transformed into any other Hamiltonian cycle K via a sequence of valid double-switch moves, such that every intermediate graph remains a Hamiltonian cycle. •Moreover, assuming $n \ge m$, the number of required moves is bounded by mn^2.•

Acknowledgments

This paper is based on part of the author's PhD dissertation at York University, written under the supervision of Professor Neal Madras.

The author is grateful to his advisor, Neal Madras, for his patience and guidance through several rewrites, and for his thoughtful and insightful feedback on each of them. The author also thanks Nathan Clisby for creating an interactive online tool for visualizing and experimenting with Hamiltonian paths and cycles[3].

This research was supported in part by a Discovery Grant from NSERC Canada to the author's advisor, Neal Madras.

Introduction

An $m \times n$ *grid graph* is the induced subgraph of the square lattice whose vertex set consists of all integer grid points $\{(i, j) : 0 \le i < m,\ 0 \le j < n\}$ with edges between vertices at distance 1. We call the unit-square faces of the square lattice *boxes*. A Hamiltonian path (cycle) of a graph G is a path (cycle) that visits each vertex of the graph exactly once.

The question of whether an $m \times n$ grid graph has a Hamiltonian path was first studied by Itai et al. in [5]. They showed that for an $m \times n$ grid graph to have a Hamiltonian cycle, it is necessary and sufficient that at least one of m and n is even. Chen et al. gave an efficient algorithm to construct Hamiltonian paths in rectangular grid graphs [2]. A *solid* grid graph is a grid graph without holes, i.e. each bounded face of the graph is a box. Umans and Lenhart [17] gave a polynomial-time algorithm to find a Hamiltonian cycle in solid grid graphs, if one exists.

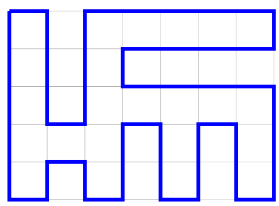

Fig. I.1. A 1-complex Hamiltonian cycle on an 8×6 grid graph.

Given any two Hamiltonian cycles H and K, the reconfiguration problem asks whether it is possible to transform H into K step-by-step, so that each intermediate step is also a Hamiltonian cycle of G. Nishat and Whitesides [13] introduced the "flip" and "transpose" moves described below, and a complexity measure called "bend complexity" for Hamiltonian cycles in rectangular grid graphs. Roughly, a 1-complex Hamiltonian cycle is one in which every vertex of G is connected to the boundary via a straight line. They prove that using these two moves, it is possible

to reconfigure any pair of 1-complex Hamiltonian cycles in G into one another. • Equivalently, the reconfiguration graph of 1-complex Hamiltonian cycles in rectangular grid graphs is connected.•
We dispense with the need for bend complexity constraints, proving the connectivity of reconfiguration graph of all Hamiltonian cycles in rectangular grid graphs, by using a more general move, which we call a double-switch move.

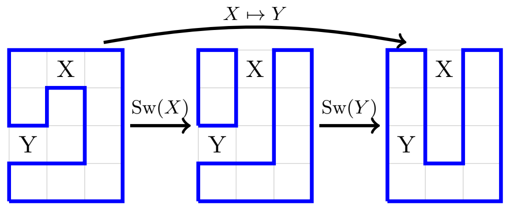

Fig. I.2. Illustration of each switch of a general double-switch move in a 4×5 grid graph.

Let H be a Hamiltonian cycle of an $m \times n$ grid graph G. A box in G with vertices a, b, c, d is considered *switchable* in H if it has exactly two edges in H, and these edges are parallel. Let $abcd$ be a switchable box with edges ab and cd in H. We define a *switch move* on the box $abcd$ in H as follows: remove edges ab and cd and add edges bc and ad. If X is a switchable box of in H, we denote a switch move on X by Sw(X).

A *double-switch move* (or simply a *move*) is a pair of switch operations where we first switch X and then find another switchable box Y and switch it, and denote it by $X \mapsto Y$. If, after a double-switch move, we obtain a new Hamiltonian cycle, we call the move a *valid move.*

Let $X = abcd$ and $Y = dcef$ be boxes sharing the edge cd of G. Assume that the edges ab, fd, dc and ce belong to H, and that the edges fe, ad and bc do not. A *flip* move consists in removing the edges fd, ce and ab, and adding the edges ad, bc and fe. Effectively, this is the same as first switching X, and then switching Y. See Figure I.3.

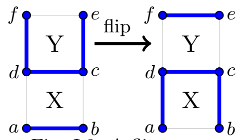

Fig. I.3. A flip move.

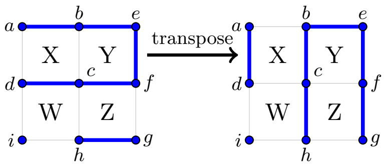

Fig. I.4. A transpose move.

Consider the four boxes $X = abcd$, $Y = cbef$, $Z = cfgh$ and $W = dchi$ that are incident on the vertex c. Note that X and Y share the edge cb, Y and Z share cf, Z and W share ch, and W and X share cd. Assume that the edges ab, be, ef, fc, cd and hg belong to H and that the edges ad, bc, ch and fg do not. A *transpose* move consists in switching X and then switching Z. See Figure I.4.

Nishat in [10] showed that flip and transpose moves are always valid. The more general double-switch moves are sufficient for constructing algorithms that reconfigure arbitrary Hamiltonian cycles in grid graphs. This comes at the added cost of verifying the validity of each move. We provide such reconfiguration algorithms and prove the existence of all required moves.

Theorem. Let H and K be any two Hamiltonian cycles in an $m \times n$ grid graph G with $n \geq m$. Then there exists a sequence of at most n^2m valid double-switch moves that reconfigures H into K.

See [1] for an illustration. • In particular, this yields an explicit mn^2 upper bound on the diameter of the reconfiguration graph of Hamiltonian cycles in $m \times n$ grid graphs•. An analogous result for Hamiltonian paths is treated in [7]. The extension makes use of two additional moves beyond the double-switch: the switch move and the *backbite* move, the latter used to relocate the endpoints of a path and originally introduced by Mansfield [9]. For a description of the backbite move, see Appendix A.2.

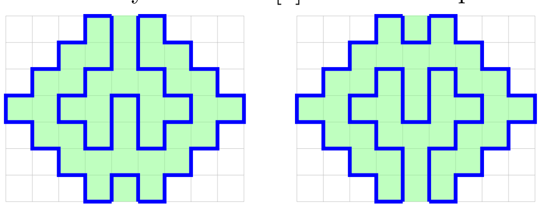

Figure I.5. A solid grid graph shaded green, with two distinct Hamiltonian cycles frozen under the double-switch move.

Scope and extensions. In Section 4 we will use the condition that the boundary is rectangular to prove the theorem. This assumption cannot be relaxed to include all general solid grid graphs. For example, neither Hamiltonian cycle in Figure I.5. admits a valid double-switch move, so it is not possible to reconfigure one into the other through double-switch moves. While we believe there may be finer classes of graphs between rectangular grid graphs and solid grid graphs that can be reconfigured by the double-switch move, it seems likely that such classes would require imposing boundary conditions on general solid grid graphs.

We conjecture that the double-switch move should also suffice to reconfigure Hamiltonian cycles in three-dimensional rectangular grid graphs, although we do not yet have a proof. By a three-dimensional rectangular grid graph we mean the induced subgraph of the cubic lattice whose vertex set is $\{(i, j, k) : 0 \leq i < m,\ 0 \leq j < n,\ 0 \leq k < p\}$ with edges between vertices at distance one. The arguments in this paper rely on Jordan's curve theorem, which has no direct analogue in three dimensions. Thus, if this conjecture is true, the proof would seem to require different techniques.

Applications and related work. A self-avoiding walk is a walk in a lattice where every vertex is unique. A Hamiltonian path in a grid graph is an example of a self-avoiding walk. Madras and Slade in [8] present a comprehensive and rigorous study of self-avoiding walks. One application of the theorem is in chemical physics, drawing from the theory of self-avoiding walks. Researchers in [16], [6], [4], and [9] use Monte Carlo methods to study statistical properties of polymer chains, which they abstracted as cycles and paths in the cubic lattice.

They use self avoiding-walks to model how a flexible polymer chain is arranged in a liquid solution. A polymer chain's concentration is the fraction of vertices of the lattice that are occupied by the vertices (monomers) of the polymer. The authors consider maximally concentrated polymers (high density polymers), where all the space is occupied by the polymer. These can be naturally represented as Hamiltonian paths or cycles. They view the set of Hamiltonian cycles in a rectangular grid graph as the state space of a Markov chain, with the double-switch move being the transition mechanism. Given a Hamiltonian cycle (a state in the state space), we choose two switchable boxes at random and perform a double-switch move. If the move is valid, then the new state is the resulting Hamiltonian cycle. Otherwise, we remain at the initial state and choose another pair of switchable boxes. The idea is that after a sufficiently large number of transitions, we obtain a set with many different states, which represents a reasonable random sample of the entire state space. The validity of these methods requires uniform random sampling of the entire state space, which in turn requires the Markov chain to be irreducible (i.e., any Hamiltonian cycle can be reconfigured into any other through a sequence of valid moves). The authors in [16, 6, 4, 9] assume irreducibility, but do not prove it. For a more detailed discussion on Monte Carlo methods and reconfiguration of self-avoiding walks, see Chapter 9 in [8].

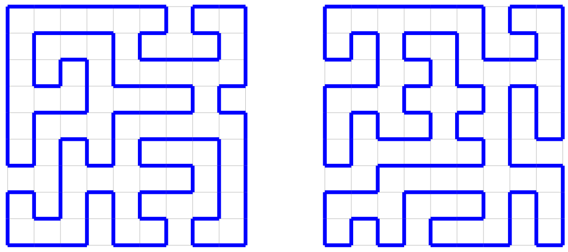

Figure I.6. Two distinct Hamiltonian cycles on 10×10 grid graphs.

Nishat, Whitesides, and Srinivasan extended the result of [13] to 1-complex Hamiltonian paths in rectangular grid graphs [12, 11, 15], and to 1-complex Hamiltonian cycles in L-shaped grid graphs [14]. The authors define a 1-complex s,t Hamiltonian path to be a 1-complex Hamiltonian path that begins and ends at diagonally opposite corners s and t of a rectangular grid graph. We note that the results [12], [11], and [15] are extended to arbitrary s,t Hamiltonian paths in [7].

The rest of the paper is organized as follows. In Section 1 we introduce notation and definitions, prove some lemmas about the structure that a Hamiltonian cycle imparts on a grid graph, and some other lemmas characterizing the validity of double-switch moves. In Section 2, we state the algorithm required for the proof of the main result, and show that it depends on the existence of a further two algorithms, the MLC and the 1LC. In Section 3 we prove the MLC and 1LC algorithms. The 1LC proof depends on a lemma whose proof takes up all of Section 4.

1 Preliminaries

A *grid graph* is a subgraph of the integer grid $\mathbb{Z}^2$. A *lattice animal* is a finite connected subgraph of $\mathbb{Z}^2$. A Hamiltonian path (cycle) of a graph G is a path (cycle) that visits each vertex of the graph exactly once. Assume that G has a cut vertex v. Then G cannot have a Hamiltonian cycle. Let G_1, G_2 be the components of $G \setminus v$. Let H_1 be a Hamiltonian path of $G \setminus G_1$ and let H_2 be a Hamiltonian path of $G \setminus G_2$ such that H_1 and H_2 have v as an end-vertex. Then a Hamiltonian path H of G can be obtained by concatenating H_1 and H_2. Since H_1 and H_2 are smaller than H they are easier to find and to reconfigure. It follows that a graph that cannot be decomposed in this manner must be 2-connected. Thus, from here on, we will restrict our attention to 2-connected grid graphs.

Definitions. Let G be a grid graph and let H be a Hamiltonian cycle of G. We denote the set of boxes of a grid graph G by Boxes(G). We will need some definitions to navigate G and H. Position G in the first quadrant so that its westernmost vertices have x-coordinate 0 and southernmost vertices have y-coordinate 0. We use the x and y coordinates to describe a rectangle in the graph and denote it by $R(k_1, k_2; l_1, l_2)$. This rectangle corresponds to the Cartesian product of the intervals (k_1, k_2) and (l_1, l_2). We will denote a box of G by $R(k, l)$ where k and l are the coordinates of the corner of the box that is closest to the origin. That is, $R(k, l) = R(k, k + 1; l, l + 1)$.

We specify a vertex v by $v(k, l)$, where k and l are the vertex coordinates. We denote a horizontal edge e by $e(k_1, k_2; l_1)$, where k_1, k_2 are the x-coordinates of the vertices of e and l_1 is the y-coordinate of the vertices of e. Similarly, we write $e(k_1; l_1, l_2)$ for vertical edges. It will be convenient to use the notation $\{u, v\}$ to describe edges of G and the notation (u, v) to describe directed edges of G. For a directed edge $e = (u, v)$, u is said to be the *tail* of e and v is said to be the *head* of e.

Let G' be a subgraph of G. Then we write $G' + (x, y)$ to denote the translation of G' by (x, y).

Theorem 1.1. Jordan's Curve Theorem for polygons (JCT). A polygon Q divides the set of points of the plane not on Q into two disjoint subsets Int (for "Interior") and Ext (for "Exterior") that have Q as a common boundary and are such that any two points within a subset can be joined by a path that does not intersect Q while any path joining a point of Int to a point of Ext must intersect Q. □

We record here a useful consequence of Jordan's curve theorem.

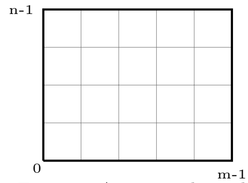

Fig. 1.1. A 6×5 grid graph.

Corollary 1.2. Let p_1 and p_2 be points on the plane not on Q. If the segment $[p_1, p_2]$ intersects Q exactly once at a point q, where q is not a vertex of Q, then one of p_1 and p_2 is on Ext and the other is on Int. □

Recall that a *solid* grid graph is a grid graph without holes. Note that an $m \times n$ grid graph is a solid grid graph such that the outer boundary is an $(m-1) \times (n-1)$ rectangle, with corners at $(0, 0), (m-1, 0), (m-1, n-1)$, and $(0, n-1)$. We call this rectangle the *boundary of* G.

Definitions. Let G be an $m \times n$ grid graph and let X_1, X_2 be two distinct boxes of G. If X_1 and X_2 share an edge of G, we say that X_1 and X_2 are adjacent. Define a *walk of boxes in* G to be a sequence $X_1, ..., X_r$ of boxes in G, not necessarily distinct, such that for all $j \in \{1, 2, ..., r-1\}$, X_j is adjacent to X_{j+1} or $X_j = X_{j+1}$, We denote such a walk by $W(X_1, X_r)$. For each $j \in \{1, ..., r-1\}$, we call the edge of G that X_j and X_{j+1} share a *gluing edge of* $W(X_1, X_r)$, whenever X_j and X_{j+1} are distinct boxes. If for all $i, j \in \{1, 2, ..., r\}$ with $i \neq j$, X_i is distinct from X_j, we call the sequence a *path of boxes in* G and denote it by $P(X_1, X_r)$. A cycle of boxes in G is a walk $X_1, ..., X_r$ such that $X_1 = X_r$ and for $i, j \in \{1, ..., r-1\}$ $X_i \neq X_j$.

•The rest of Section 1 contains definitions and technical results used in Sections 2-4. Our reconfiguration strategy relies on controlling which edges belong to the Hamiltonian cycle by applying moves. To analyze when moves can add or remove specific edges while preserving the Hamiltonian property, we need to understand how the Hamiltonian cycle H decomposes the grid graph into components that we call H-components. This decomposition is introduced in Section 1.1. In section 1.2 we define the Follow-the-wall construction and use it to build walks of boxes that respect the structure of H, called H-walks. H-walks are used extensively in many proofs throughout the rest of the paper. In Section 1.3 we prove some basic properties of H-components; In section 1.4 we give a more detailed description of double-switch moves and prove a lemma about their validity. •

1.1 The H-decomposition of G

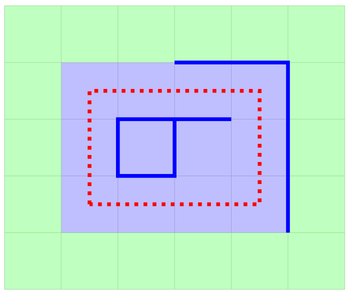
Fig. 1.2. An $m \times n$ grid graph G shaded blue, a subgraph H of G in blue, BBoxes(G) shaded green, an H-cycle in G dotted red.

Definitions. A *walk* (of length r) in a graph is an alternating sequence $v_0 e_1 v_1 e_2 .... e_r v_r$ of vertices and edges. Define a *lazy walk* to be sequence of edges and vertices where every edge is in between two vertices that are its endpoints, and in between every two edges there is a vertex or multiple copies of a vertex. That is, a lazy walk is roughly a walk in which consecutive vertices can be the same, allowing the walk to remain at a vertex for one or more steps without traversing any edges.

Let G be an $m \times n$ grid graph and let H be any subgraph in G. Let X_1, X_2 be two adjacent boxes of G. If $E(X_1) \cap E(X_2) \cap E(H) = \emptyset$, we say that X_1 and X_2 are *H-neighbours* or X_1 is *H-adjacent* to X_2. Define an *H-walk of boxes in* G (H-walk) to be a sequence $X_1, ..., X_r$ of boxes in G, not necessarily distinct, such that for all $j \in \{1, 2, ..., r-1\}$, X_j is an H-neighbour of X_{j+1} or $X_j = X_{j+1}$.

If for all $i, j \in \{1, 2, ..., r\}$ with $i \neq j$, X_i is distinct from X_j, we call the sequence an *H-path of boxes in* G and denote it by $P(X_1, X_r)$. Let $r \geq 4$. Define an *H-cycle of boxes in* G (H-cycle) to be a set $X_1, X_2, ..., X_r = X_1$ of boxes in G such that for each $j \in \{1, 2, ..., r-1\}$, X_j is an H-neighbour of X_{j+1} and the boxes $X_1, ..., X_{r-1}$ are distinct. Let C be an H-cycle of boxes in G. We note that every box of C has exactly two gluing edges. Proposition 1.3 will show that if H is a Hamiltonian cycle of G, then there are no H-cycles of boxes in G.

Define a *boundary box* of G to be a box that is incident on the boundary of G but that is not a box of G. denote the set of boundary boxes of G by *BBoxes(G)*. Define G_{-1} to be the graph with vertex set $V(G_{-1}) = V(G) \cup V(\text{BBoxes}(G))$ and edge set $E(G_{-1}) = E(G) \cup E(\text{BBoxes}(G))$. We extend the definitions of H-walks, H-paths and H-cycles to G_{-1}. See Figure 1.2 for an illustration.

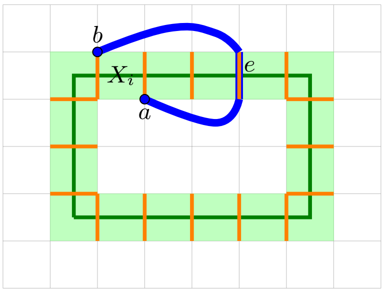

Fig. 1.3. $P(a, b)$ in blue, gluing edges of C in orange, Q in dark green, C shaded in light green.

Proposition 1.3. Let G be an $m \times n$ grid graph and let H be Hamiltonian cycle of G. Then every H-cycle in G_{-1} is contained in BBoxes(G).

Proof. By way of contradiction assume that there is an H-cycle C in G_{-1} with boxes $X_1, X_2, ..., X_r = X_1$ that has a box X_i contained in G. Let $c_1, c_2, ..., c_r = c_1$ be the centers of the boxes of C. That is, if $X_j = R(k, l)$ then $c_j = (k + \frac{1}{2}, l + \frac{1}{2})$. Note that for each $j \in \{1, 2, ..., r-1\}$, $[c_j, c_{j+1}]$ intersects the gluing edge of X_j and X_{j+1} and $[c_j, c_{j+1}]$ intersects no other edge of G_{-1}.

We will first show that the set of segments $[c_j, c_{j+1}]$ is a non-self-intersecting polygon Q. Since $c_1 = c_r$, Q is a polygon. For a contradiction assume that Q is self intersecting, so there are points c_{j-1}, c_j, c_{j+1} and c_i such that the segments $[c_{j-1}, c_j]$, $[c_j, c_{j+1}]$ and $[c_j, c_i]$ are edges of Q. But then X_j has three gluing edges, a contradiction.

By JCT, Q divides the plane into two subsets Int and Ext such that any two points within a subset can be joined by a path that does not intersect Q while a path joining a point of Int to a point of Ext must intersect Q. Let $V(\text{Int})$ be the vertices of G_{-1} contained in Int and let $V(\text{Ext})$ be the vertices of G_{-1} contained in Ext. Note that any box of C contains at least one vertex in $V(\text{Int})$ and one vertex in $V(\text{Ext})$, so both sets are nonempty. In particular, this is true for X_i. For definiteness, assume that $a \in V(X_i)$ is contained in $V(\text{Int})$ and $b \in V(X_i)$ is contained in $V(\text{Ext})$. By JCT, $V(\text{Int}) \cup V(\text{Ext}) = V(G)$ and $V(\text{Int}) \cap V(\text{Ext}) = \emptyset$. Consider the subpath $P(a, b)$ of H. By JCT again, there is an edge $e \in P(a, b) \subset H$ intersecting Q at some segment $[c_j, c_{j+1}]$. But then e is a gluing edge of C, so e cannot belong to H. □

Let G be an $m \times n$ grid graph and let H be a Hamiltonian cycle of G. Let $X_1, ..., X_r$ be a set of boxes in G such that for any $i, j \in \{1, 2, ..., r\}$, there is an H-path $P(X_i, X_j)$ between X_i and X_j contained in G. We call X_i and X_j the *end-boxes* of $P(X_i, X_j)$. We say that X_i and X_j are H-path-connected in G and the set of boxes $\{X_1, ..., X_r\}$ is an *H-path connected set of boxes in* G. If the set $\{X_1, ..., X_r\}$ contains no cycles of boxes we call it an *H-tree*. A *H-component* of G is a maximal H-path connected set of boxes.

Corollary 1.4. A Hamiltonian cycle H partitions the boxes of G into H-path-connected H-components which are maximal H-trees. In particular, if an H-path is contained in an H-component, that H-path is unique. □

1.2 The follow-the-wall construction

Definitions. Let G be a graph. A *trail* is a walk in G where all edges are distinct. A trail where the first and last vertices coincide is called a *closed trail* or a *circuit*. A *directed walk* (of length s) is an alternating sequence $v_0 e_1 v_1 e_2 .... e_s v_s$ of vertices and directed edges such that for $j \in \{1, ..., s\}$, the directed edge e_j has tail v_{j-1} and head v_j. A *directed trail* is a directed walk where all directed edges are distinct. We will use the notation $\overrightarrow{K}$ to denote directed trails.

Let the box X be incident on a directed edge (u, v) of the integer grid. Then X and its vertices not incident on the edge e_j are either on the *right* or on the *left* side of (u, v). See Figure 1.4 for an illustration and Appendix A.1 for a more precise definition of a box being on the right or left side of a directed edge. Note that if X is on the right side of (u, v), then the other box incident on (u, v), say X', is on the right side of (v, u).

Let H be a cycle in an $m \times n$ grid graph G. Let $K = v_1, ..., v_{s+1}$ be a subpath of H. We orient K to obtain a directed trail $\overrightarrow{K} = (v_1, v_2), ..., (v_s, v_{s+1}) = e_1 ...., e_s$. We will use $\overrightarrow{K}$ to construct an H-walk of boxes in G_{-1} that we will call *the right H-walk induced by* $\overrightarrow{K}$ and denote it by $W_{\overrightarrow{K},\text{right}}(X_1, X_r)$, where X_1 and X_r are the end-boxes of $W_{\overrightarrow{K},\text{right}}(X_1, X_r)$. Roughly, $W_{\overrightarrow{K},\text{right}_,}(X_1, X_r)$ will be the walk of boxes that a "walker" would encounter as they followed along the side of $\overrightarrow{K}$ when starting on the right side of the first edge e_1 of $\overrightarrow{K}$. We will call this construction the *follow-the-wall* construction (FTW). This is very similar to the well-known hand-on-the-wall maze-solving algorithm.

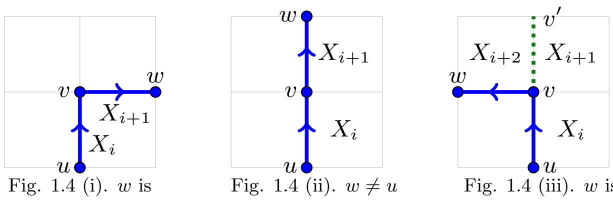

Fig. 1.4 (i). w is right of e_j. Fig. 1.4 (ii). $w \neq u$ is collinear with e_j. Fig. 1.4 (iii). w is left of e_j.

Let X be the box of G_{-1} on the right of the edge e_1 of $\overrightarrow{K}$ and let e_j be the j^{th} edge of $\overrightarrow{K}$. Then $X = X_1$ is the first box of the H-walk. Let $e_j = (u, v)$, $e_{j+1} = (v, w)$ and let X_i be on the right of the edge e_j. There are three possibilities for the position of w with respect to (u, v): w is right of (u, v), $w \neq u$ is colinear with (u, v) and w is left of (u, v). For the last case define $e'_j = (v, v')$ to be the edge in $G_{-1} \setminus H$ that is colinear with e_j and set $e''_j = (v', v)$. See Figure 1.4.

(i) *w is right of e_j.* Then X_{i+1} is on the right of the edge e_{j+1}. Note that in this case, the walk has a repeated box since $X_i = X_{i+1}$.

(ii) *$w \neq u$ is collinear with e_j.* Then X_{i+1} is on the right of the edge e_{j+1}.

(iii) *w is left of e_j.* Then X_{i+1} is on the right of the edge e'_j and X_{i+2} is on the right of the edge e_{j+1}.

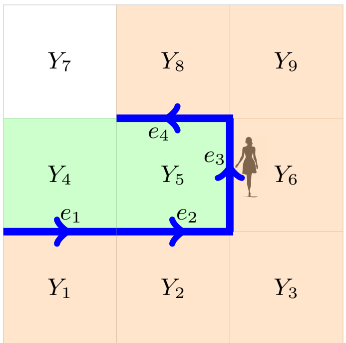

Fig. 1.5. A subtrail $\overrightarrow{K} = \overrightarrow{K}(e_1, e_4)$ of a Hamiltonian cycle of a grid graph G in blue; $\Phi(\overrightarrow{K}, \text{right}) = Y_1Y_2Y_3Y_6Y_9Y_8$ shaded orange and $\Phi(\overrightarrow{K}, \text{left}) = Y_4Y_5Y_5Y_5$ shaded green; silhouette in gray following the wall along e_3.

We say that the edge e_{j+1} *adds* to the H-walk $W_{\overrightarrow{K},\text{right}}(X_1, X_r)$ the box X_{i+1}, in cases (i) and (ii), and boxes X_{i+1} and X_{i+2} in case (iii). If e_s adds more than one box to $W_{\overrightarrow{K},\text{right}}(X_1, X_r)$ then we will adopt the convention that X_r is the last box added by the edge e_s and that all the boxes added by each edge of e_j, $j \in \{1, ..., s\}$ are on the right side of e_j. Note that this convention is necessary for the box X_{i+1} in Case (iii). We remark that the first edge e_1 can only add the single box X_1. The left H-walk $W_{\overrightarrow{K},\text{left}}(X_1, X_r)$ induced by $\overrightarrow{K}$ can be constructed analogously.

Let H denote either a path or a cycle in G. Let $\mathcal{K}(H)$ be the set of all directed subtrails of H. We can view the FTW construction as a function Φ that assigns an H-walk to elements of $\mathcal{K}(H) \times \{\text{right}, \text{left}\}$. We will take a closer look at H-trails in the case where H is a Hamiltonian cycle of G.

Let $H = v_1, ..., v_r, v_1$ be a Hamiltonian cycle in G. Orient H as a directed circuit $\overrightarrow{K}_H$. We observe that it is possible to choose a starting edge $e_j = (v_j, v_{j+1})$ of $\overrightarrow{K}_H$ so that the directed circuit $\overrightarrow{K}_H = e_j, e_{j+1}, ..., e_{j-1}$ is such that $\text{Boxes}(\Phi(\overrightarrow{K}_H, \text{right})) \cup \text{Boxes}(\Phi(\overrightarrow{K}_H, \text{left})) = \text{Boxes}(G_{-1})$. We record an equivalent statement for reference as Observation 1.5 below. From here on, all circuits $\overrightarrow{K}_H$ will be assumed to satisfy Observation 1.5.

Let $H = v_1, v_2, ..., v_r, v_1$ be a Hamiltonian cycle in G. Note that any subtrail of $\overrightarrow{K}_H$ is completely determined by its first and last edges. Therefore, it will be fitting to use the notation $\overrightarrow{K}(e_s, e_t)$ to denote the unique subtrail starting at edge e_s and ending at edge e_t.

1.3 Properties of H-components

Observation 1.5. Let G be an $m \times n$ grid graph and let H be a Hamiltonian cycle of G. Then for every box X in G_{-1} there is an edge e of $\overrightarrow{K}_H$ and side $\in \{\text{right, left}\}$ such that e adds X to $W_{\overrightarrow{K}_H,\text{ side}}$.

Lemma 1.6. Let G be an $m \times n$ grid graph and let $Q = v_1, ..., v_r, v_1$ be any cycle in G and let U be the region bounded by Q. Orient the edges of G into the directed circuit $\overrightarrow{K}_Q = (v_1, v_2), ..., (v_r, v_1)$. Then $\text{Boxes}(\Phi(\overrightarrow{K}_Q, \text{right})) \subseteq U$ and $\text{Boxes}(\Phi(\overrightarrow{K}_Q, \text{left})) \subseteq G_{-1} \setminus U$ iff $\Phi((v_1, v_2), \text{right})$ is a box of U.

Proof. This follows from Corollary 1.2, the definition of FTW, and induction on the edges of $\overrightarrow{K}_Q$. □

Note that, by JCT, H divides the boxes of G_{-1} into the disjoint sets $\text{int}(H)$ and $\text{ext}(H)$, where $\text{int}(H)$ is the bounded region.

Proposition 1.7. Let G be an $m \times n$ grid graph and let H be a Hamiltonian cycle of G. Then $\text{int}(H)$ is an H-component of G.

Proof. Let $H = v_1, ..., v_r, v_1$ and, for definiteness, assume that $\Phi((v_1, v_2), \text{right}) \in \text{int}(H)$. By Lemma 1.6, $\text{Boxes}(\Phi(\overrightarrow{K}_H, \text{right})) \subseteq \text{int}(H)$ and $\text{Boxes}(\Phi(\overrightarrow{K}_H, \text{left})) \subseteq G_{-1} \setminus \text{int}(H) = \text{ext}(H)$. Since H is contained in G and H is the boundary of $\text{int}(H)$, $\text{int}(H)$ is contained in G. We check that $\text{int}(H)$ is H-path-connected and maximal.
Consider any box $Z \in \text{int}(H)$. By Observation 1.5, there is an edge $e \in \overrightarrow{K}_H$ such that e adds Z to $\Phi(\overrightarrow{K}_H, \text{left})$ or e adds Z to $\Phi(\overrightarrow{K}_H, \text{right})$. The former implies that Z belongs to $\text{ext}(H)$, contradicting

that $Z \in \text{int}(H)$. It must be the case that $Z \in \text{Boxes}(\Phi(\overrightarrow{K}_H, \text{right}))$, which is H-path connected. Note that this also shows that $\text{int}(H) = \text{Boxes}(\Phi(\overrightarrow{K}_H, \text{right}))$.
To see that $\text{int}(H)$ is maximal, we note that $G \subset \text{int}(H) \cup \text{ext}(H)$ and that $\text{int}(H) \cap \text{ext}(H) = \emptyset$, so $\text{int}(H)$ cannot be extended. □

Corollary 1.8. Let G be an $m \times n$ grid graph, $H = v_1, ..., v_r, v_1$ be a Hamiltonian cycle of G, and assume that $\Phi((v_1, v_2), \text{right})$ is a box of $\text{int}(H)$. Then $\text{Boxes}(\Phi(\overrightarrow{K}_H, \text{right})) = \text{int}(H)$ and $\text{Boxes}(\Phi(\overrightarrow{K}_H, \text{left})) = \text{ext}(H)$. □
Let G be an $m \times n$ grid graph and let H be a Hamiltonian cycle of G. Let $J_0, J_1, ..., J_s$ be the H-components of G, where $J_0 = \text{int}(H)$. It follows from Proposition 1.7 that $\text{BBoxes}(G) = G_{-1} \setminus G$ is contained in $\text{ext}(H)$ and that all other components $J_1, ..., J_s$ of G are contained in $\text{ext}(H) \setminus \text{BBoxes}(G)$. We write this as Observation 1.10 below for reference. We call the H-components $J_1, ..., J_s$ *cookies of G*. If a cookie J has more than one box, we call J a *large* cookie. Otherwise, we say that J is a *small* cookie.

Let J be a cookie of G. We define a *neck of J* to be a box N_J of J that is incident on a boundary edge e_J of G with $e_J \notin H$. We call e_J a *neck-edge* of J. Note that the other box incident on e_J must be in $G_{-1} \setminus G$. With these definitions, Lemma 1.6 has the following corollary:

Corollary 1.9. Let G be an $m \times n$ grid graph and let H be a Hamiltonian cycle of G, and let $J_0, ..., J_s$ be the H-components of G. Then every edge of H is incident on a box of J_0 and a box of $G_{-1} \setminus J_0$. □

Observation 1.10. Let J be an H-component of G. Then $J \subseteq \text{int}(H)$ or $J \subset \text{ext}(H)$.

Corollary 1.11. Let X and Y be boxes of an H-component J. Then X and Y are on the same side of $\overrightarrow{K}_H$.

Lemma 1.12. Let G be an $m \times n$ grid graph, let H be a Hamiltonian cycle of G, and let J be a cookie of G. Then J has a unique neck.

Proof. If J only has one box, we are done, so assume that J has more than one box. Let Z be a box of J. By Observation 1.5, we may assume, WLOG, that Z is on the left of $e_z \in \overrightarrow{K}_H$. If no edge is incident on Z, we choose one of the four neighbours beside it.
We claim that there exists a subtrail $\overrightarrow{K}(e_z, e_{t+1})$ of $\overrightarrow{K}_H$ such that $\Phi\big(\overrightarrow{K}(e_z, e_t), \text{left}\big)$ is contained in J but $\Phi\big(\overrightarrow{K}(e_z, e_{t+1}), \text{left}\big)$ is not. Assume for contradiction that for every subtrail of $\overrightarrow{K}_H$ starting at e_z, $\Phi\big(\overrightarrow{K}(e_z, e_j), \text{left}\big)$ is contained in J, where $j \in \{z+1, ..., z\}$. But then $\Phi\big(\overrightarrow{K}_H, \text{left}\big)$ is contained in $J \subset \text{Boxes}(G)$, contradicting that $\Phi\big(\overrightarrow{K}_H, \text{left}\big)$ contains the boxes of $G_{-1} \setminus G$. It follows that e_{t+1} adds the first box Y of $\Phi\big(\overrightarrow{K}(e_z, e_{t+1}), \text{left}\big)$ that is not contained in J. Note that, by definition of H-component, Y must belong to $G_{-1} \setminus G$. (Since Y is H-adjacent to the box X preceding it, but Y does not belong to J, it must be the case that Y is not in G). Let X be the box of J preceding Y in $\Phi\big(\overrightarrow{K}(e_z, e_{t+1}), \text{left}\big)$. We have that X and Y are H-adjacent and share a boundary edge e_J of G that is not in H. By definition of neck of an H-component, $X = N_J$.

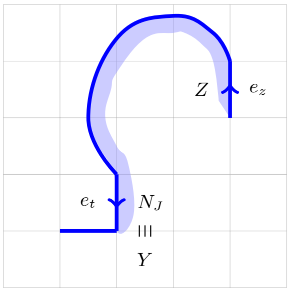

Fig. 1.6. $\Phi(\overrightarrow{K}(e_z, e_t), \text{left})$ shaded in blue.

To see that the neck of J is unique, assume for contradiction that J has at least two necks, N_1 and N_2. By Corollary 1.8, $\text{ext}(H)$ is H-path-connected. Let N_1' and N_2' be the boxes in $\text{BBoxes}(G)$ that are adjacent to N_1 and N_2, respectively. Then there is an H-cycle $P(N_1', N_2')$, $P(N_2, N_1)$ in G_{-1} that is not contained in $\text{BBoxes}(G)$, which contradicts Proposition 1.3. □

Lemma 1.13. Let G be an $m \times n$ grid graph, let H be a Hamiltonian cycle of G, and let J be a cookie of G with neck N_J. Orient H and let (v_x, v_{x+1}) and (v_{y-1}, v_y) be edges of N_J in H, where $\{v_x, v_y\}$ is a boundary edge of G. Then $V(J) = V(\overrightarrow{K}((v_x, v_{x+1}), (v_{y-1}, v_y)))$.

Proof. Let $\overrightarrow{K}_1 = \overrightarrow{K}((v_x, v_{x+1}), (v_{y-1}, v_y))$ and $\overrightarrow{K}_2 = \overrightarrow{K}((v_y, v_{y+1}), (v_{x-1}, v_x))$, and note that $\overrightarrow{K}_H = \overrightarrow{K}_1, \overrightarrow{K}_2$. For definiteness, assume that $N_J = \Phi((v_x, v_{x+1}), \text{right})$.
Let $v_z \in V(J)$. Assume for contradiction that $v_z \notin \overrightarrow{K}_1$. Then $v_z \in \overrightarrow{K}_2$. Let Z be a box of J on which v_z is incident, and let e_z be the edge of $\overrightarrow{K}_2$ containing v_z that adds Z to $\Phi(\overrightarrow{K}_H, \text{right})$. Note that we used Corollary 1.11 here. By proof of Lemma 1.12, there is a subtrail $\overrightarrow{K}(e_z, e_t)$ of $\overrightarrow{K}_H$ such that $\Phi(\overrightarrow{K}(e_z, e_t), \text{right})$ is contained in J but $\Phi(\overrightarrow{K}(e_z, e_{t+1}), \text{right})$ is not contained in J. (If no edge of H is

incident on Z we can just consider $\overrightarrow{K}(e_{z-1}, e_t)$ instead.) Note that this implies that e_t must add N_J to $\Phi(\overrightarrow{K}(e_z, e_t), \text{right})$. Then $e_t = (v_x, v_{x+1})$ or $e_t = (v_{y-1}, v_y)$.

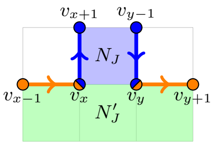

Fig. 1.7. $G_{-1} \setminus G$ shaded in green.

Let N'_J be the neighbour of N_J in $G_{-1} \setminus G$. Note that $N'_J \in \Phi(\overrightarrow{K}(e_z, (v_x, v_{x+1})), \text{right})$ and so $\Phi(\overrightarrow{K}(e_z, (v_x, v_{x+1}))$ is not contained in J. Then it must be the case that $e_t = (v_{y-1}, v_y)$.

Since $e_z \in \overrightarrow{K}_2$ and $e_t = (v_{y-1}, v_y) \in \overrightarrow{K}_1$, there must be some $j_0 \in \{z, z+1, ..., y-2\}$ such that for each $j \leq j_0$, $e_j \in \overrightarrow{K}_2$, but $e_{j_0+1} \in \overrightarrow{K}_1$. It follows that $e_{j_0+1} = (v_x, v_{x+1})$ or $e_{j_0+1} = (v_{y-1}, v_y)$.

Note that if $e_{j_0+1} = (v_{y-1}, v_y)$ then $(v_{y-2}, v_{y-1}) = e_{j_0}$. But $(v_{y-2}, v_{y-1}) \in \overrightarrow{K}_1$, contradicting $e_{j_0} \in \overrightarrow{K}_2$. Then it must be the case that $e_{j_0+1} = (v_x, v_{x+1})$. But then $\Phi(\overrightarrow{K}(e_z, e_{j_0}), \text{right})$ contains $\Phi(e_{j_0}, \text{right})$, which belongs to $G_{-1} \setminus G$, contradicting that $\Phi(\overrightarrow{K}(e_z, e_{j_0}), \text{right})$ is contained in J. Thus we must have that $v_z \in \overrightarrow{K}_1$.

It remains to check that $V(\overrightarrow{K}_1) \subseteq V(J)$. We will prove that $\overrightarrow{K}_1 \subseteq E(J)$. For $i \in \{x, x+1, ..., y-1\}$, let $e_i = (v_i, v_{i+1})$. Note that if $\overrightarrow{K}(e_x, e_i) \subseteq E(J)$ then $\Phi(\overrightarrow{K}(e_x, e_i), \text{right}) \subset J$. This follows by definition of FTW and induction on the edges of $\overrightarrow{K}(e_x, e_i)$.

We have that $e_x \in \overrightarrow{K}_1 \cap E(J)$. Assume for contradiction that there exists some $j_0 \in \{x+1, x+2, ..., y-3\}$[1] such that $\overrightarrow{K}(e_x, e_{j_0})$ is contained in $E(J)$ but $\overrightarrow{K}(e_x, e_{j_0+1})$ is not. Since $e_{j_0+1} \notin E(J)$, we have that $\Phi(\overrightarrow{K}(e_x, e_{j_0}), \text{right})$ is contained in J but $\Phi(\overrightarrow{K}(e_x, e_{j_0+1}), \text{right})$ is not. Let Z be the first box of $\Phi(\overrightarrow{K}(e_x, e_{j_0+1}), \text{right})$ that is not contained in J and let Z' be the box preceding Z in $\Phi(\overrightarrow{K}(e_x, e_{j_0+1}), \text{right})$. The fact that $Z \in J$ and $Z' \notin J$ are H-neighbours implies that $Z' = N_J$. Since e_{j_0+1} adds Z to $\Phi(\overrightarrow{K}(e_x, e_{j_0+1}), \text{right})$, $e_{j_0+1} = e_x$ or $e_{j_0+1} = (v_y, v_{y+1})$. But both possibilities contradict that $j_0 \in \{x+1, x+2, ..., y-3\}$. □

Let H be a subgraph of G. A box of G on vertices a, b, c, d is *switchable* in H if it has exactly two edges in H and the edges are parallel to each other. •We call e box of G with exactly three edges in H a *leaf*.•

Lemma 1.14. Let G, H and $J_0, ..., J_s$ be as in Corollary 1.9. Then a large cookie has exactly one box incident on a boundary edge of G, namely its neck. Furthermore, the neck of each large cookie is switchable.

Proof. First we show that a large cookie has exactly one box in incident on a boundary edge of G, namely its neck. Let J_i be a cookie. By Lemma 1.12, J_i has a neck N_{J_i} and N_{J_i} is incident on the boundary of G. Suppose that there is another box X of J_i that is incident on B_0. Note that $X \in G_{-1} \setminus J_0$. Let e be the boundary edge of X and let Y be the box in $G_{-1} \setminus G$ that is incident on e. Then $e \in H$ or $e \notin H$. Note that $e \notin H$ contradicts Lemma 1.12 so we only need to check the case where $e \in H$. Suppose that $e \in H$. By Corollary 1.9, $Y \in J_0 \subset G$, contradicting that $Y \in G_{-1} \setminus G$.

Now we show that the neck of each large cookie is a switchable box. Let $X = R(k, l)$ be the neck of a cookie J_i. Let $v(k, l) = a$, $v(k+1, l) = b$, $v(k+1, l+1) = c$ and $v(k, l+1) = d$. For definiteness, assume that $\{a, b\}$ is the neck edge of J_i. Observe that we must have $0 < k < m-1$. It follows that $\{a, d\} \in H$ and $\{b, c\} \in H$. Since J_i is not a small cookie $\{c, d\} \notin H$. Thus, X is switchable. □

1.4 Moves

Definitions. Let G be an $m \times n$ grid graph and let H be a Hamiltonian cycle of G. Let $abcd$ be a switchable box with edges ab and cd in H. A *switch move* on the box $abcd$ in H removes edges ab and cd and adds edges bc and ad. Let $X \in G$ be a switchable box in H. We write $\text{Sw}(X)$ to denote a switch move.

A *double-switch move* is pair of switch moves where we first switch X and then find a switchable Y and switch it. We denote a double-switch move by $X \mapsto Y$. See Figure 1.9. If after a double-switch, we get a new Hamiltonian cycle, then we call the move a *valid move*. We call $X \mapsto X$ a *trivial* move.

Orient H as directed cycle $v_1, ..., v_r, v_1$. Let X be a switchable box in H with edges $e_1 = (v_s, v_{s+1})$ and $e_2 = (v_t, v_{t+1})$. If v_s is adjacent to v_t in G, we say that e_1 and e_2 are *parallel*; and if v_s is adjacent to v_{t+1} in G, we say that e_1 and e_2 are *anti-parallel*. Similarly, we call the box X a *parallel (anti-parallel)* box if its edges are parallel (anti-parallel).

[1] $\Phi(e_{y-1}, \text{right}) = N_J$, so $j_0 \leq y-3$

Let J be a large cookie with neck N_J. We call a valid flip move $N_J \mapsto N'_J$ a neck-shifting flip (NSF) move. Observe that after a NSF move, the large cookie J necessarily becomes the large cookie $(J \cup N'_J) \setminus N_J$, with new neck N'_J.

We define a *cascade* to be a sequence of moves $\mu_1, ..., \mu_r$ such that for $0 \leq j \leq r-1$:

1) μ_1 is valid,
2) if $\mu_1, ..., \mu_j$ have been applied then μ_{j+1} is valid, and
3) the sequence may contain NSF moves but does not otherwise create any new cookies.

Let H be a Hamiltonian e-cycle of an $m \times n$ grid graph G and let J be a cookie of H with neck N_J . Consider a cascade $\mu_1, ..., \mu_r$ where μ_r is the nontrivial move $Z \mapsto N_J$. We say that the cascade $\mu_1, ..., \mu_r$ *collects* the cookie J. Note that all double-switch moves are invertible. For non-adjacent boxes X and Y , the moves $X \mapsto Y$ and $Y \mapsto X$. When X and Y are adjacent with X switchable and Y a leaf (i.e. $X \mapsto Y$ is a flip move), X must be switched first before Y becomes switchable, so the order matters.

Lemma 1.15. Let G be an $m \times n$ grid graph and let H be a Hamiltonian cycle of G. Orient H as a directed cycle $v_1, ..., v_r, v_1$. Then every switchable box of H is anti-parallel.

Proof. Assume for contradiction that there is a box X in H with parallel edges e_1 and e_2. For definiteness assume that $X = R(k,l)$, $e_1 = e(k, k+1; l)$, $e_2 = e(k, k+1; l+1)$ and that $\text{Boxes}(\Phi(\overrightarrow{K}_H, \text{right})) = \text{int}(H)$. But then $X = \Phi(e_2, \text{right}) \in \text{int}(H)$ and $\Phi(e_1, \text{right}) \in \text{int}(H)$, contradicting Corollary 1.9. □

We will show below that if we switch a switchable box of H we get a cycle H_1 and a cycle H_2. We define a (H_1, H_2)*-port* to be a switchable box of $H_1 \cup H_2$ that has one edge in H_1 and the other in H_2.

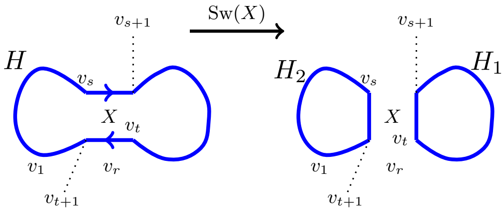

Fig. 1.8 (a). A directed Hamiltonian cycle H, X is switchable, e_1 and e_2 are anti-parallel.

Fig. 1.8 (b). Cycles H_1 and H_2, obtained from H after $Sw(X)$.

Lemma 1.16. Let G be be an $m \times n$ grid graph and let H be a Hamiltonian cycle of G. Orient H as a directed cycle $v_1, ..., v_r, v_1$. Let X be a switchable box of H with edges $e_1 = (v_s, v_{s+1})$ and $e_2 = (v_t, v_{t+1})$, with $s+1 < t$.

(i) Sw(X) splits H into two cycles, H_1 and H_2.

(ii) Suppose we apply Sw(X). If Y is an (H_1, H_2)-port then $X \mapsto Y$ is a valid double-switch move.

Proof. Removing edges e_1 and e_2 splits H into two disjoint paths $P_1 = P(v_{s+1}, v_t)$ and $P_2 = P(v_{t+1}, v_r)$, $\{v_r, v_1\}, P(v_1, v_s) = P(v_{t+1}, v_s)$. By Lemma 1.15, e_1 and e_2 are anti-parallel. Then we have that v_s is adjacent to v_{t+1} and v_{s+1} is adjacent to v_t. Now ; adding $e'_1 = (v_{s+1}, v_t)$ gives a cycle $H_1 = P_1, e'_1$; and adding $e'_2 = (v_s, v_{t+1})$ gives a cycle $H_2 = P_2, e'_2$. End of proof for (i).

The proof of (ii) is essentially the same as the proof for (i), so we omit it. □

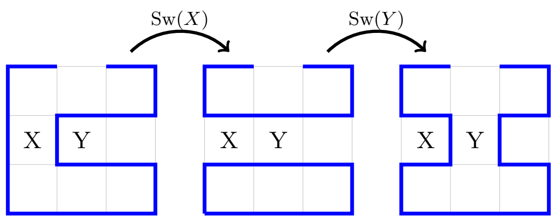

Fig. 1.9 (a). A Hamiltonian e-cycle H on a 4×4 grid; X is switchable.

Fig. 1.9 (b). H_{cycle} and H_{path} after switching X. Note that Y is an (H_{cycle}, H_{path})-port.

Fig. 1.9 (c). A Hamiltonian e-cycle H' after switching Y.

Hamiltonian e-cycles. Let G be an $m \times n$ grid graph, let H be a Hamiltonian cycle of G and let e be an edge of H that lies in the boundary of G. We call the path $H' = H \setminus e$ a Hamiltonian *e-cycle* of G. We remark that all the definitions and results about the case where H is Hamiltonian cycles of G translate immediately to the case where H' is a Hamiltonian e-cycle of G. We may just add back the edge e incident on the end-vertices of the H' to obtain the cycle H. All relevant properties we have observed remain unchanged.

Section 2 contains algorithms we can use to reconfigure one Hamiltonian cycle (e-cycle) into another. Proofs of existence for the algorithms are in Section 3. Section 4 contains proofs of auxiliary results required in Section 3.

1.5 Appendix

A.1. Let $A = (x_1, y_1)$, $B = (x_2, y_2)$ and $P = (x, y)$ be points in the plane that are not collinear. We define $(x_2 - x_1, y_2 - y_1)$ as the direction of the vector $\overrightarrow{AB}$. Then the direction of the normal $\overrightarrow{n}$ to $\overrightarrow{AB}$, obtained by rotating $\overrightarrow{AB}$ by $-\frac{\pi}{2}$, is $(y_2 - y_1, x_1 - x_2)$. We want to know whether the

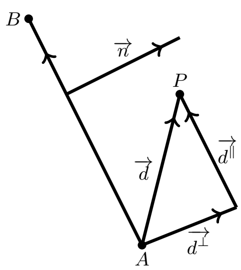

Fig. 1.10. P on the right of $\overrightarrow{AB}$.

point P is on the side of $\overrightarrow{AB}$ toward which $\overrightarrow{n}$ is pointing. Let $\overrightarrow{d} = \overrightarrow{AP}$. Let $\overrightarrow{d^{\perp}}$ be the component of $\overrightarrow{d}$ that is perpendicular to $\overrightarrow{AB}$ and let $\overrightarrow{d^{\parallel}}$ be the component of $\overrightarrow{d}$ that is parallel to $\overrightarrow{AB}$. Note that:

$$\overrightarrow{d}\cdot\overrightarrow{n} = (\overrightarrow{d^{\parallel}}+\overrightarrow{d^{\perp}})\cdot\overrightarrow{n} = \overrightarrow{d^{\perp}}\cdot\overrightarrow{n} = (x-x_1, y-y_1)\cdot(y_2-y_1, x_1-x_2) = (x-x_1)(y_2-y_1)+(y-y_1)(x_1-x_2).$$

Point P is on the *right* of $\overrightarrow{AB}$ if $\overrightarrow{d}\cdot\overrightarrow{n} > 0$, and on the *left* if $\overrightarrow{d}\cdot\overrightarrow{n} < 0$. Let $e = (u, v)$ be an edge of a grid graph G, where $u = v(k_1, l_1)$, $v = v(k_2, l_2)$. Let X be a box of the square lattice that is incident on (u, v). We say that X is on the *right* of the edge (u, v) if there is a vertex $w = v(k, l)$ in $V(X) \setminus V(e)$ such that $(k - k_1, l_2 - l_1) + (l - l_1, k_1 - k_2) = 1$ and we say that X is on the *left* of the edge (u, v) if there is a vertex $w = v(k, l)$ in $V(X) \setminus V(e)$ such that $(k - k_1, l_2 - l_1) + (l - l_1, k_1 - k_2) = -1$.

A.2. Let H be a Hamiltonian path $v_1, \dots, v_r$ of an $m \times n$ grid graph G, and let v_s be adjacent to v_1, $s \neq 2$. If we add the edge $\{v_1, v_s\}$, we obtain a cycle $v_1, \dots, v_s, v_1$, and a path $v_s, \dots, v_r$. Now, if we remove the edge $\{v_{s-1}, v_s\}$, we obtain a new Hamiltonian path $H' = (H \setminus \{v_{s-1}, v_s\}) \cup \{v_1, v_s\}$. This operation is called a backbite move. See Figure 1.11.

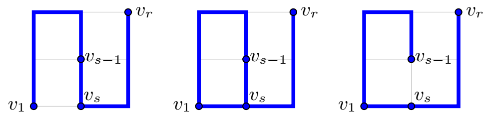

Fig. I.11. An illustration of a backbite move.

2 Reconfiguration algorithm for cycles and canonical forms

Definitions. Denote by G_s the induced subgraph of G on all the vertices with distance s or greater from the boundary of G. Denote by R_s the rectangular induced subgraph on vertices of G with distance s from the boundary. Then R_s is the boundary of G_s and the edges of R_0 are the boundary edges of G.

The main result stated in the Introduction is an immediate consequence of the slightly more general theorem below. Its proof takes up the remainder of the paper.

Theorem 2.1. Let G be an $m \times n$ grid graph with $n \geq m$. Let H and K be two Hamiltonian cycles or Hamiltonian e-cycles of G with the same edge e. Then there is a sequence of at most n^2m valid double-switch moves that reconfigures H into K.

The $3 \times n$ and $4 \times n$ cases were done by Nishat in [10], so from here on, we will assume that $m, n \geq 5$, and that m and n are not both odd. First we will describe canonical forms for Hamiltonian cycles and e-cycles. Then we show that we can reconfigure any two canonical forms into one another. Then we show that any Hamiltonian cycle (e-cycle) can be reconfigured into a canonical form. Observing that double-switch moves are invertible completes the proof. That is $X \mapsto Y$ followed by $Y \mapsto X$ results in no net change. More specifically, suppose we want to reconfigure a Hamiltonian cycle (e-cycle) H into a Hamiltonian cycle (e-cycle) K. Let $\mu_1, ..., \mu_k$ and $\nu_1, ..., \nu_s$ be the sequences of moves that reconfigure H and K into the canonical forms H_{can} and K_{can}, respectively. Then $\nu_s, \nu_{s-1}, ..., \nu_1$ reconfigures K_{can} into K. Let $\eta_1, ..., \eta_t$ be the sequence of moves that reconfigures H_{can} into K_{can}. Then the sequence of moves $\mu_1, ..., \mu_k, \eta_1, ..., \eta_t, \nu_s, \nu_{s-1}, ..., \nu_1$ reconfigures H into K.

Description of canonical forms. We shall write $\mathcal{H}_{\text{can}}(m, n)$ to denote the set of canonical forms of Hamiltonian cycles and e-cycles on an $m \times n$ grid graph. Then $H \in \mathcal{H}_{\text{can}}(m, n)$ if and only if H can be constructed by the "Canonical Form Builder" algorithm described below.

Let $t = \left\lfloor \frac{\min(m,n)-4}{2} \right\rfloor$. Let $k_1 = |m - n| + 2$ and $k_2 = |m - n| + 3$. If $\min(m, n)$ is even, let D be the Hamiltonian cycle of the $2 \times k_1$ grid graph G_{t+1}. If $\min(m, n)$ is odd, let D be any Hamiltonian cycle of the $3 \times k_2$ grid graph G_{t+1}. Let $U = D \cup \bigcup_{i=0}^{t} R_i$.

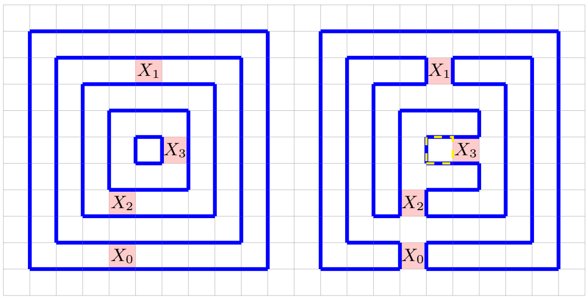

Fig. 2.1(a). U with $\min(m,n)$ even. D_1 in dashed yellow.

Fig. 2.1(b), A canonical form from U.

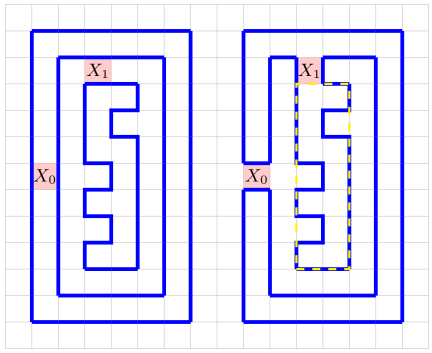

Fig. 2.2(a). U with $\min(m,n)$ odd.

Fig. 2.2(b). A canonical form from U. $D_2(H)$ in dashed yellow.

We define (R_i, R_{i+1}) to be the set of all the boxes of G adjacent to both R_i and R_{i+1}. Now we can state the Canonical Form Builder algorithm (CFB) that takes as inputs m and n and outputs an element of $\mathcal{H}_{\text{can}}(m,n)$.

Step 1. Set $i = 0$. Switch one of the $2(m-3)+2(n-3)$ switchable boxes of (R_0, R_1) of the graph U. This switch removes some edge, say e_1, from $E(R_1)$. If $t = 0$, stop. If $t > 0$, go to Step 2.

Step 2. Increase i by 1. Switch one of the switchable boxes of $(R_i, R_{i+1}) \setminus e_i$.

Step 3. If $i \leq t$, go to Step 2. If $i = t+1$, then stop.

We have arrived at a canonical form H. Record the switched boxes $X_0, ..., X_t$ in a list *List(H)*. So List(H) $= (X_0, ..., X_t)$ consists of the faces of G that were chosen to be switched to make U into a canonical form, listed in order.

We observe that the CFB algorithm above works just as well for e-cycles if we remove e from U.

Reconfiguration between canonical forms. Let $H, K \in \mathcal{H}_{\text{can}}(m,n)$. Let List(H) $= (X_0, ..., X_t)$ and List(K) $= (Y_0, ..., Y_t)$ be the switched boxes of H and K respectively. We will reconfigure H into K, so the algorithm will run on H.

Let $D(H) = H \cap G_{t+1}$ and note that $D(H)$ is an e-cycle of G_{t+1}. Using the result of Nishat in [10], $D(H)$ can be reconfigured into $D(K)$ by a sequence of valid moves.

Step 1. Set $i = 0$. If $t = 0$, go to Step 3. If $t > 0$, go to Step 2.

Step 2. If Y_i is switchable after switching X_i, switch both X_i and Y_i.

If Y_i is not switchable after switching X_i, switch X_{i+1} and any other switchable box in (R_{i+1}, R_{i+2}), say X'_{i+1}, such that $X_{i+1} \mapsto X'_{i+1}$ is valid. We remark that the only the only case where Y_i would not be switchable after switching X_i occurs when Y_i is adjacent to X_{i+1}. Note that there are many possible choices for X'_{i+1}. Now Y_i is switchable. Switch both X_i and Y_i. Update List(H) by setting the $(i+1)^{st}$ slot to X'_{i+1}. Increase i by 1.

Step 3. If $i < t$, go to Step 2.

If $i = t$ and $\min(m,n)$ is even, switch X_i and Y_i, and then stop.

If $i = t$ and $\min(m,n)$ is odd, go to Step 3.1.

Step 3.1. Switch X_t and any one of the four switchable boxes, say X'_t, located on the short sides of D. Run NRI's algorithm to reconfigure $D(H)$ into $D(K)$. Switch X'_t and Y_t. Stop.

Reconfiguration of a cycle into a canonical form (RtCF). The RtCF algorithm takes as input a Hamiltonian e-cycle and outputs a canonical e-cycle. We will need the following proposition:

Proposition 2.2. Let $H \in \mathcal{H}$.

(a) If H has more than one large cookie, then there is a cascade of length at most two that reduces the number of large cookies of H by one. This is the ManyLargeCookies (MLC) algorithm.

(b) If H has exactly one large cookie and at least one small cookie, then there is a cascade of length at most $\frac{1}{2}\max(m,n) + \min(m,n) + 2$ that reduces the number of small cookies of H by one and such that it does not increase the number of large cookies. This is the OneLargeCookie (1LC) algorithm.

The proof for Proposition 2.2 will be given in Section 3.

Now we can describe the RtCF algorithm. Without loss of generality, assume $H \in \mathcal{H}$ is a Hamiltonian e-cycle of G. Suppose $H = H_0$ has $c_{1;\text{large}}$ large cookies and $c_{1;\text{small}}$ small cookies. We run MLC $c_{1;\text{large}} - 1$ times and then run 1LC $c_{1;\text{small}}$ times to reconfigure H_0 into H_0', where H_0' has exactly one (necessarily large) cookie C_1. We define $H_1 = (G_1 \cap H_0')$ and observe that H_1 is a Hamiltonian e-cycle of G_1. This is the first iteration of (RtCF). Now we describe the j^{th} iteration. We run MLC $c_{j;\text{large}} - 1$ times and then run 1LC $c_{j;\text{small}}$ times to reconfigure H_{j-1} into H_{j-1}', where H_{j-1}' has exactly one (necessarily large) cookie C_j. The RtCF algorithm stops when $j = t$. We give a summary of the algorithm below.

Step 1. Set $j = 0$. Run MLC $c_{1;\text{large}}$ times and then 1LC $c_{1;\text{small}}$ times on H_0 to reconfigure H_0 into H_0'.

Step 2. Increase j by 1. Set $H_j = G_j \cap H_{j-1}'$ and note that H_j is a Hamiltonian e-cycle in G_j. Run MLC $c_{j+1;\text{large}}$ times and then 1LC $c_{j+1;\text{small}}$ times on H_j to reconfigure into H_j into H_j'.

Step 3. If $j < t$, go to Step 2. If $j = t$, stop.

Proof of the RtCF algorithm. Let N_j be the neck of the only cookie C_j of H_{j-1}' in G_{j-1}. Define $e_1(N_j) = N_j \cap R_{j-1}$, $e_2(N_j) = N_j \cap R_j$ and $\{e_3(N_j), e_4(N_j)\} = N_j \cap H_{j-1}'$. We observe that when the RtCF algorithm stops, we have reconfigured H into

$$H_c = D(H) \cup \bigcup_{j=0}^{t} \big(R_j \cup e_3(N_{j+1}) \cup e_4(N_{j+1})\big) \setminus (e_1(N_{j+1}) \cup e_2(N_{j+1}).$$

Now we can see that H_c is an element of $\mathcal{H}_{\text{can}}(m,n)$ by setting $X_{j-1} = N_j$ for $j = 1, 2, ..., t+1$ and running CFB. See Figure 2.3 on page 12 for an illustration of the RtCF algorithm on a 10×10 grid.

Bound for Theorem 2.1. Recall that $n \geq m$. Note that it takes at most $2m$ moves to reconfigure canonical forms into one another. Now we count the moves required for RtCF to terminate. Observe that for each $j \in \{0, \dots, t-1\}$, H_j has at most $2\big(\frac{n-2j}{2} + \frac{m-2j}{2}\big) = n + m - 4j$ cookies. This is the number of iterations of MLC or 1LC required for each j. It will follow from the proofs in Sections 3 and 4 that each application of MLC or 1LC in H_j requires at most $\frac{1}{2}n + m - 3j + 3$ moves. So, RtCF requires at most:

$$\begin{aligned}
&\sum_{j=1}^{\lfloor (m-2)/2 \rfloor} (n+m-4j)\Big(\frac{n}{2} + m - 3j + 2\Big) \\
&= \sum_{j=1}^{\lfloor (m-2)/2 \rfloor} \Big(12j^2 + (-5n - 7m - 8)j + \frac{n^2}{2} + \frac{3nm}{2} + 2n + m^2 + 2m\Big) \\
&\leq \tfrac{1}{2}(m^3 - 3m^2 + 2m) + (-5n - 7m - 8)\Big(\frac{m^2}{8} - \frac{m}{4}\Big) + \tfrac{1}{2}\Big(\frac{n^2}{2} + \frac{3nm}{2} + 2n + m^2 + 2m\Big)(m-2) \\
&= \frac{n^2 m}{4} + \frac{nm^2}{8} + \frac{3nm}{4} + \frac{m^3}{8} - \frac{3m^2}{4} - \frac{n^2}{2} + m - 2n \\
&= \frac{n^2 m}{2} + \frac{nm}{4}\Big(3 - \frac{3m}{n} - \frac{2n}{m}\Big) + m - 2n.
\end{aligned}$$

Let $x = \frac{m}{n}$. Then $\frac{3m}{n} + \frac{2n}{m} = 3x + \frac{2}{x}$. Using calculus, we find that it attains a minimum of $2\sqrt{6}$ at $x = \frac{\sqrt{6}}{3}$. Then $\big(3 - \frac{3m}{n} - \frac{2n}{m}\big)$ can be at most $3 - 2\sqrt{6} \leq -1$. It follows that RtCF requires at most $\frac{n^2 m}{2} - \frac{nm}{4} + m - 2n$ to terminate. For a complete reconfiguration we need to run RtCF once for each e-cycle and reconfigure the resulting canonical forms. So, we need at most $2\big(\frac{n^2 m}{2} - \frac{nm}{4} + m - 2n\big) + m = n^2 m - \frac{nm}{2} - 4n + 3m < n^2 m$ moves for a complete reconfiguration. We remark that this is a worst case scenario and conjecture that the typical number of moves required is of the order of n^2.

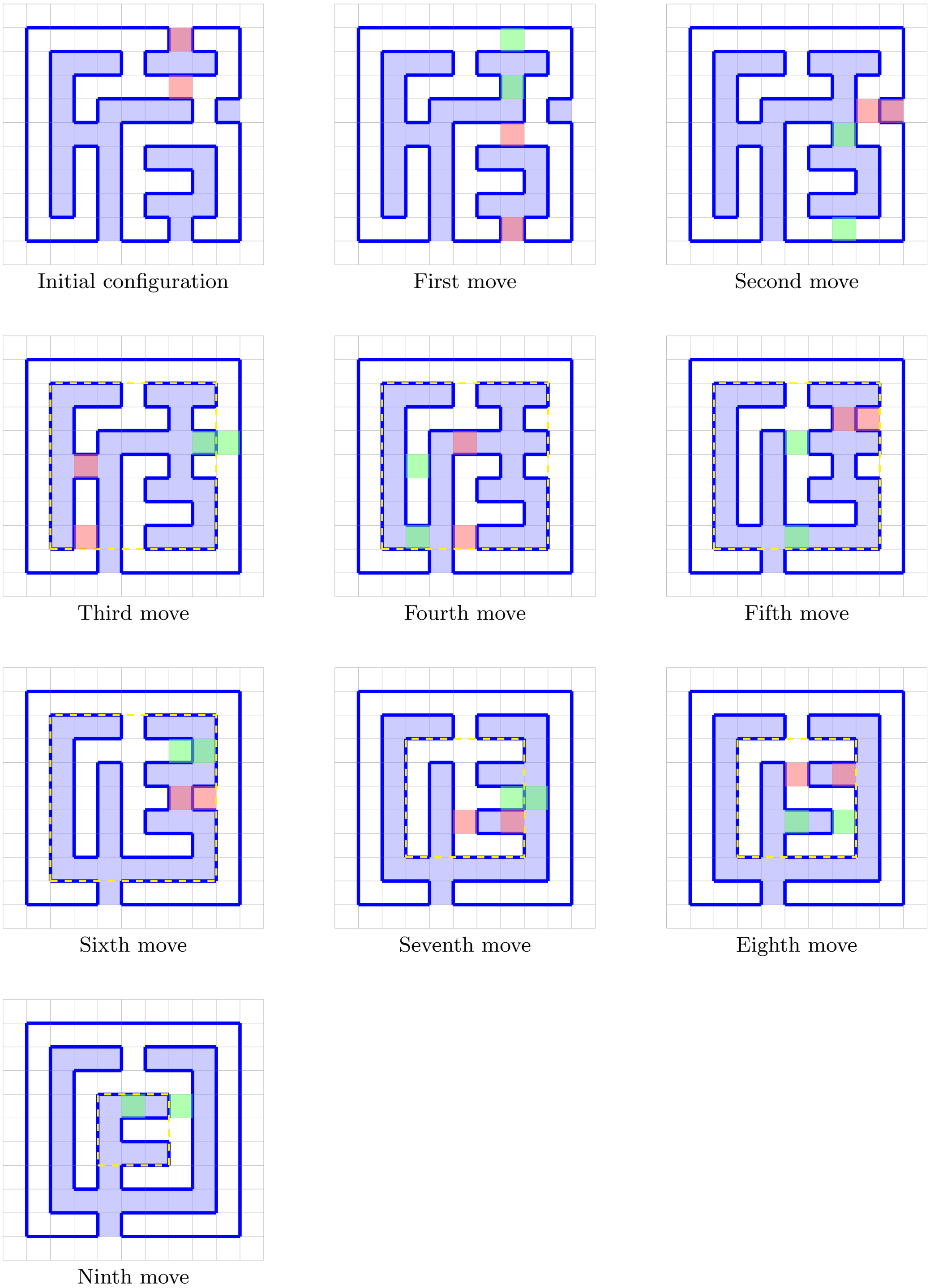

Figure 2.3. An illustration of RtCF on a 10×10 grid graph. Pink squares indicate boxes that have just been switched; green squares indicate boxes that we'll be switched next. The first three moves are in the $j = 0$ iteration of RtCF, the next three moves are in the $j = 1$ iteration, and the last three moves are in the $j = 2$ iteration. No moves are required for the $j = 3$ iteration, since H_3 has exactly one large cookie.

3 Existence of the MLC and 1LC Algorithms

• Recall that RtCF algorithm in Section 2 presupposes the existence of the moves required for its execution. The proofs of existence were deferred to the ManyLargeCookies (MLC) and OneLargeCookie (1LC) algorithms, which we prove in Sections 3.1 and 3.2, respectively. These algorithms ensure that whenever RtCF requires a particular move, either the move is immediately available, or else there exists a cascade after which the required move becomes available. Importantly, such cascades do not undo the progress already made: RtCF does not regress. The restrictions in the definition for cascades at the end of Section 1.4 were designed precisely for this reason.

Consider an iteration of RtCF on rectangle G_i with Hamiltonian e-cycle H_i. At this stage, H_i either has more than one large cookie, exactly one large cookie with at least one small cookie, or exactly one large cookie with no small cookies. In the last case, H_i is already in the desired form for this iteration, and RtCF proceeds to G_{i+1}. The MLC algorithm handles the first case by finding the required cascades to collect large cookies when multiple large cookies are present. The 1LC algorithm handles the second case by finding the required cascades to collect small cookies when exactly one large cookie remains.

Why do we need two separate algorithms for what appears to be the same task? This is because small cookies can be harder to collect than large ones. A second large cookie J always has a switchable neck N_J; to collect J we need only find another switchable box X such that $N_J \mapsto X$ is a valid move, or a cascade delivering such a switchable box. In Section 3.1, we show that it takes at most two moves to accomplish this (Proposition 3.8). Small cookies, by contrast, consist of a single non-switchable box. To collect a small cookie, either the box Y adjacent to it in (R_i, R_{i+1}) must be switchable, or we must find a cascade that makes Y switchable. The latter task can require much longer cascades, and it is more difficult to deal with. It requires Lemma 3.7, all of Section 3.2, most of Section 4, and several results from Chapter 1. Furthermore, the assumption that exactly one large cookie is present significantly shortens and simplifies the proofs of Proposition 3.10 and Lemmas 3.11–3.15 in Section 3.2, by precluding the possibility of several tedious cases. •

3.1 Existence of the MLC Algorithm

Definitions. Let G be an $m \times n$ grid graph, let H be a Hamiltonian cycle of G, and let W be a switchable box in H. Let X and Y be the boxes adjacent to W that are not its H-neighbours, and assume that X and Y belong to the same H-component. By Corollary 1.4, the H-path $P(X, Y)$ is unique. We call $P(X, Y)$ *the looping H-path of W*. See Figure 3.1 for an illustration of the looping H-path of a switchable box W in a Hamiltonian cycle of a 6×6 grid graph.

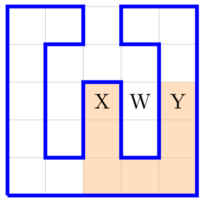

Fig. 3.1. The looping H-path of W shaded orange.

Outline of the MLC algorithm. Let H be a Hamiltonian cycle of an $m \times n$ grid graph G with multiple large cookies. We first identify a large cookie J with switchable neck N_J. Consider what happens if N_J is switched: this would produce two cycles, H_1 and H_2. First we observe that there must be some edge $\{v_1, v_2\}$ in R_2 (recall the nested rectangles from Chapter 2) with $v_1 \in H_1$ and $v_2 \in H_2$ (Lemma 3.7). The proximity of $\{v_1, v_2\}$ to the boundary constrains the possible configurations of edges in its vicinity. We analyze those configurations (Lemma 3.5) and show that either an (H_1, H_2)-port already exists near $\{v_1, v_2\}$, or a single-move cascade on the original H yields a Hamiltonian cycle H' where such a port exists after switching N_J.

Proposition 3.1. Let G be an $m \times n$ grid graph, let H be a Hamiltonian cycle of G, and let $P(X, Y)$ be the looping H-path of a switchable box W. Let H' be the graph consisting of the cycles H_1 and H_2 obtained after switching W. Then a box Z of G belongs to the H'-cycle $P(X, Y), W, X$ if and only if Z is incident on a vertex of H_1 and on a vertex of H_2.

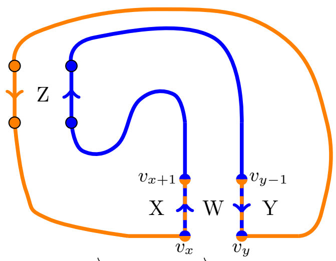

Fig. 3.2. $\overrightarrow{K}_1$ in blue, $\overrightarrow{K}_2$ in orange.

Proof. Orient H as a directed cycle $H = v_1, ..., v_r, v_1$. By Lemma 1.15, W is anti-parallel. Let the edges of W in H be $\{v_x, v_{x+1}\}$ and $\{v_{y-1}, v_y\}$. For definiteness, assume that X is adjacent to $\{v_x, v_{x+1}\}$, Y is adjacent to $\{v_{y-1}, v_y\}$ and that W is on the right of $\{v_x, v_{x+1}\}$. Then we have that $\Phi((v_x, v_{x+1}), \text{left}) = X$ and $\Phi((v_{y-1}, v_y), \text{left}) = Y$. Define $\overrightarrow{K}_1$ and $\overrightarrow{K}_2$ to be the subtrails $\overrightarrow{K}((v_x, v_{x+1}), (v_{y-1}, v_y))$ and $\overrightarrow{K}((v_{y-1}, v_y), (v_x, v_{x+1}))$ of $\overrightarrow{K}_H$, respectively. By Lemma 1.16 (i), switching W splits H into two cycles H_1 and H_2, with $V(H_1) = V(\overrightarrow{K}_1) \setminus \{v_x, v_y\}$ and $V(H_2) = V(\overrightarrow{K}_2) \setminus \{v_{x+1}, v_{y-1}\}$.

Proof of ($\Longrightarrow$). Since $P(X,Y)$ is unique, any H-walk of boxes between X and Y contains $P(X,Y)$. In particular, $\Phi(\overrightarrow{K}_1,\text{left})$ contains $P(X,Y)$ and $\Phi(\overrightarrow{K}_2,\text{left})$ contains $P(X,Y)$. Let Z be a box of $P(X,Y)$. Then Z is added to $\Phi(\overrightarrow{K}_1,\text{left})$ by an edge of $\overrightarrow{K}_1$ and Z is added to $\Phi(\overrightarrow{K}_2,\text{left})$ by an edge of $\overrightarrow{K}_2$. By definition of FTW, Z is incident on a vertex of $\overrightarrow{K}_1$ and a vertex of $\overrightarrow{K}_2$. Since for $i \in \{1,2\}$, $V(\overrightarrow{K}_i) \supset V(H_i)$, we have that Z is incident on a vertex of H_1 and a vertex of H_2. Lastly, note that W is incident on $v_{x+1} \in H_1$ and $v_x \in H_2$. End of proof for ($\Longrightarrow$).

Proof of ($\Longleftarrow$). Suppose we switch W and obtain the graph H' consisting of the cycles H_1 and H_2. Observe that $P(X,Y), W, X$ is the only H'-cycle in G. We will say that a box Z of G satisfies $(*)$ if Z is incident on a vertex in H_1 and H_2. We will show that if a box Z of G satisfies $(*)$ then it must belong to an H'-cycle of boxes that satisfy $(*)$. Then, since there is only one H'-cycle in G, $Z = W$ or $Z \in P(X,Y)$.

Let Z be a box in G that satisfies $(*)$. For definiteness, assume that $Z = R(k,l)$. We will show that Z has exactly two neighbours in G that satisfy $(*)$ and that Z is H'-adjacent to those two neighbours. Since Z satisfies $(*)$, either Z has two vertices in H_1 and two vertices in H_2 or Z has one vertex in one of H_1 and H_2 and three vertices in the other.

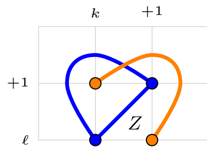

Fig. 3.3.

CASE 1: Z has two vertices in H_1 and two vertices in H_2. First we will check that the pair of vertices belonging to H_i, $i \in \{1,2\}$ must be adjacent in Z. Assume for contradiction that $v(k,l)$ and $v(k+1,l+1)$ belong to H_1 and $v(k+1,l)$ and $v(k,l+1)$ belong to H_2. See Figure 3.3. Let Q be the closed polygon consisting of the subpath $P(v(k,l), v(k+1,l+1))$ of H_1 and the segment $[v(k,l), v(k+1,l+1)]$. Then, by Theorem 1.1, $v(k+1,l)$ and $v(k,l+1)$ are on different sides of H_1. It follows that the subpath $P(v(k+1,l), v(k,l+1))$ of H_2 intersects Q. Since $P(v(k+1,l), v(k,l+1))$ does not intersect Q at the segment $[v(k,l), v(k+1,l+1)]$, it must intersect Q at some vertex in $P(v(k,l), v(k+1,l+1))$. But this contradicts that H_1 and H_2 are disjoint. It follows that the pair of vertices belonging to H_i, $i \in \{1,2\}$ must be adjacent in Z.

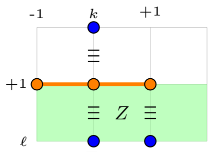

Fig. 3.4.

For definiteness assume that $v(k,l)$ and $v(k+1,l)$ belong to H_1 and that $v(k+1,l+1)$ and $v(k,l+1)$ belong to H_2. See Figure 3.4. Since H_1 and H_2 are disjoint $e(k;l,l+1) \notin H'$ and $e(k+1;l,l+1) \notin H'$ so $Z+(-1,0)$ and $Z+(1,0)$ are H'-adjacent to Z. Since $v(k,l) \in H_1 \cap V(Z+(-1,0))$ and $v(k,l+1) \in H_2 \cap V(Z+(-1,0))$, $Z+(-1,0)$ satisfies $(*)$. Similarly, $Z+(1,0)$ satisfies $(*)$. It remains to check that $Z+(0,1)$ and $Z+(0,-1)$ do not satisfy $(*)$.
Assume for contradiction that one of $Z+(0,1)$ and $Z+(0,-1)$ satisfies $(*)$. By symmetry we may assume WLOG that $Z+(0,1)$ satisfies $(*)$. Then at least one of $v(k,l+2)$ and $v(k+1,l+2)$ belongs to H_1. By symmetry we may assume WLOG that $v(k,l+2) \in H_1$. It follows that $e(k;l+1,l+2) \notin H'$ and that $v(k-1,l+1) \in H_2$. Then we must have $e(k-1,k;l+1) \in H_2$ and $e(k,k+1;l+1) \in H_2$. But then, by Corollary 1.2, one of $v(k,l)$ and $v(k,l+2)$ belongs inside the region bounded by H_2 and the other belongs outside it. It follows that the subpath $P(v(k,l), v(k,l+2))$ of H_1 intersects H_2, contradicting that H_1 and H_2 are disjoint. End of Case 1.

CASE 2: Z has one vertex in one of H_1 and H_2 and three vertices in the other. For definiteness assume that $v(k,l)$, $v(k,l+1)$ and $v(k+1,l+1)$ belong to H_1 and that $v(k+1,l)$ belongs to H_2. Then $e(k,k+1;l) \notin H'$ and $e(k+1;l,l+1) \notin H'$, so $Z+(1,0)$ and $Z+(0,-1)$ are H'-neighbours of Z. Since $v(k,l) \in H_1 \cap V(Z+(0,-1))$ and $v(k+1,l) \in H_2 \cap V(Z+(0,-1))$, $Z+(0,-1)$ satisfies $(*)$. Similarly, $Z+(1,0)$ satisfies $(*)$. It remains to check that $Z+(0,1)$ and $Z+(0,-1)$ do not satisfy $(*)$.

Assume for contradiction that one of $Z+(-1,0)$ and $Z+(0,1)$ satisfies $(*)$. By symmetry we may assume WLOG that $Z+(0,1)$ satisfies $(*)$. Then one of $v(k,l+2)$ and $v(k+1,l+2)$ belongs to H_2. Note that if $v(k+1,l+2) \in H_2$ we run into the same contradiction as in Case 1, so we only need to check the case where $v(k,l+2) \in H_2$. Now, either $e(k,k+1;l+1) \in H_1$, or $e(k,k+1;l+1) \notin H'$.

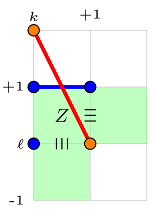

Fig. 3.5(a).

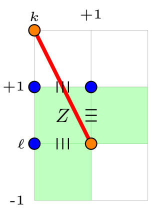

Fig. 3.5(b).

CASE 2.1: $e(k,k+1;l+1) \in H_1$. Then the segment $[v(k,l+2), v(k+1,l)]$ intersects H_1 at the point $(k\frac{1}{2}, l+1)$. by Corollary 1.1, the vertices $e(k,l+2)$ and $v(k+1,l)$ are on different sides of H_1, and we run into the same contradiction as in Case 1 again. End of Case 2.1. See Figure 3.5(a)

CASE 2.2: $e(k,k+1;l+1) \notin H_1$. Consider the polygon Q consisting of the segment $[v(k,l+2), v(k+1,l)]$ and the subpath $P(v(k,l+2), v(k+1,l))$ of H_2. By Corollary 1.1 the vertices $v(k,l+1)$ and $v(k+1,l+1)$ are on different sides of Q. BY JCT the subpath $P(v(k,l+1), v(k+1,l+1))$ of H_1 intersects Q. Since $P(v(k,l+1), v(k+1,l+1))$ does not intersect Q at the segment $[v(k,l+2), v(k+1,l)]$, it must do so at some vertex of $P(v(k,l+2), v(k+1,l))$, contradicting that H_1 and H_2 are disjoint. See Figure 3.5(b). End of Case 2.2. End of Case 2. End of proof for ($\Longleftarrow$). □

Corollary 3.2 (i). Let G be an $m \times n$ grid graph, let H be a Hamiltonian cycle of G, and let W be a switchable box in H. Let H' be the graph consisting of the cycles H_1 and H_2 obtained after switching W. Let a, b and c be colinear vertices such that b is adjacent to a and c. Then, for $i \in \{1,2\}$, If a and c belong to H_i, so must b. See Figure 3.4. □

Corollary 3.2 (ii). Let G be an $m \times n$ grid graph, let H be a Hamiltonian cycle of G, and let W be a switchable box in H. Let H' be the graph consisting of the cycles H_1 and H_2 obtained after switching W. Let Z be a box on vertices a, b, c, and d such that a and b belong to H_1, and c and d belong to H_2. Then a is adjacent to b, and c is adjacent to d. See Figure 3.3. □

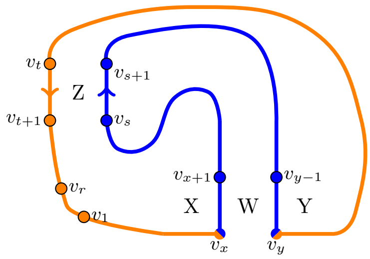

Fig. 3.6. P_1 in blue, P_2 in orange.

Proposition 3.3. Let H be a Hamiltonian cycle of an $m \times n$ grid graph G, let W be a switchable box in H and let $P(X,Y)$ be the looping H-path of W. If $P(X,Y)$ has a switchable box Z, then $Z \mapsto W$ is a valid move.

Proof. Let $H = v_1, v_2 ..., v_r, v_1$. Let W, $P(X,Y)$, $\{v_x, v_{x+1}\}$ and $\{v_{y-1}, v_y\}$ be as in Proposition 3.1. By Lemma 1.15, W is anti-parallel. Let $P_1 = P(v_x, v_y)$ and let $P_2 = P(v_y, v_x)$. By Proposition 3.1, every box of $P(X,Y)$ is incident on a vertex of P_1 and a vertex of P_2. Let Z be a switchable box of $P(X,Y)$. Let (v_s, v_{s+1}) and (v_t, v_{t+1}) be the edges of Z in H. For definiteness, assume $s+1 < t$. Proposition 3.1 implies that exactly one of (v_s, v_{s+1}) and (v_t, v_{t+1}) is in P_1 and the other is in P_2. WLOG assume that (v_s, v_{s+1}) is in P_1 and that (v_t, v_{t+1}) is in P_2. Then we can partition the edges of H as follows: $P(v_1, v_s)$, (v_s, v_{s+1}), $P(v_{s+1}, v_t)$, (v_t, v_{t+1}), $P(v_{t+1}, v_r)$, $\{v_r, v_1\}$ where $1 < x < s < y < t < r$.

We we check that $Z \mapsto W$ is a valid move. After removing the edges (v_s, v_{s+1}) and (v_t, v_{t+1}) we are left with two paths: $P(v_{t+1}, v_s)$ and $P(v_{s+1}, v_t)$. Note that adding the edge $\{v_s, v_{t+1}\}$ gives a cycle H_1 consisting of the path $P(v_s, v_{t+1})$ and the edge $\{v_s, v_{t+1}\}$. and adding the edge $\{v_{s+1}, v_t\}$ gives a cycle H_2 consisting of the path $P(v_{s+1}, v_t)$ and the edge $\{v_{s+1}, v_t\}$. Now $1 < x < s$ implies that $(v_x, v_{x+1}) \in H_1$ and $s < y < t$ implies that $(v_{y-1}, v_y) \in H_2$. It follows that W is now an (H_1, H_2)-port. By Lemma 1.16 (ii), $Z \mapsto W$ is a valid move. □

Observation 3.4. Let $X \mapsto Y$ be a valid move. If $X \in \text{ext}(H)$ and $Y \in \text{int}(H)$ then:

(i) $X \mapsto Y$ increases the total number of cookies if and only if $X \in G_1$ and $Y \in G_0 \setminus G_1$.

(ii) $X \mapsto Y$ increases the total number of large cookies, leaving the total number of cookies unchanged, if and only if $X \in G_1$, $Y \in G_1 \setminus G_2$ and Y is adjacent to a small cookie.

(iii) $X \mapsto Y$ decreases the total number of cookies if and only if $X \in G_0 \setminus G_1$ and $Y \in G_1$.

Lemma 3.5. Let H be a Hamiltonian cycle of an $m \times n$ grid graph G and let Z be a switchable box in $Z \in \text{ext(H)} \cap ((G_0 \setminus G_1) \cup G_3)$. Assume that switching Z splits H into the cycles H_1 and H_2 that are such that there is $v_1 \in H_1 \cap R_2$ and $v_2 \in H_2 \cap R_2$ with v_1 adjacent to v_2. Then either $Z \mapsto Z'$ is a cascade, or there is a cascade $\mu, Z \mapsto Z'$ (of length two), with $Z \mapsto Z'$ nontrivial in either case.

In figures 3.7 through 3.11, vertices and edges of H_1 and H_2 are in blue and orange, respectively, and boxes of int(H) are shaded in green.

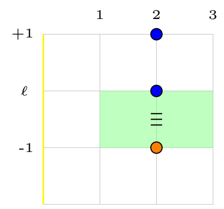

Fig. 3.7 (a).

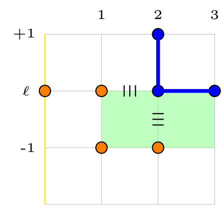

Fig. 3.7 (b). Case 1.

Proof. We will use the assumption that $Z \in (G_0 \setminus G_1) \cup G_3$ repeatedly and implicitly throughout the proof. Switch Z to obtain H' consisting of the disjoint cycles H_1 and H_2. Note that now, if a vertex belongs to H_i for $i \in \{1,2\}$, both edges incident on it must also belong to H_i. For definiteness, let $v_1 \in H_1 \cap R_2$ be the vertex $v(2,l)$ for some $l \in \{2, ..., n-3\}$ and let $v_2 = v(2, l-1) \in H_2 \cap R_2$. Then $e(2; l-1, l) \notin H$, and by Corollary 3.2 (i), $v(2, l+1) \in H_1$ as well. By Proposition 3.1, $R(1, l-1)$ and $R(2, l-1)$ belong to int(H). Now, either $v(1,l) \in H_1$, or $v(1,l) \in H_2$. See Figure 3.1.

CASE 1: $v(1,l) \in H_2$. Then $e(1,2;l) \notin H$. Corollary 3.2(i), $v(3,l) \in H_1$. It follows that $e(2;l,l+1) \in H'$ and $e(2,3;l) \in H'$. Now, by Corollary 3.2 (ii), $v(1,l-1) \in H_2$ and by Corollary 3.2 (i), $v(0,l) \in H_2$. At this point we must either have, $e(1; l, l+1) \in H'$ or $e(1; l, l+1) \notin H'$.

CASE 1.1: $e(1; l, l+1) \notin H'$. Then $e(1; l-1, l) \in H_2$ and $e(0, 1; l) \in H_2$. Since $Z \in \text{ext(H)}$, by Proposition 3.1, $R(1, l-1) \in \text{int(H)}$. Then $R(0, l-1)$ must be a small cookie of G, so we must have that $e(0, 1; l-1) \in H_2$ and that $e(1, 2; l-1) \notin H'$. It follows that $e(2, 3; l-1) \in H'$ and $e(3; l-1, l) \notin H'$. Now note that $R(2, l-1)$ is an (H_1, H_2)-port. Then, by Lemma 1.16 (ii) and Observation 3.4, $Z \mapsto R(2, l-1)$ is valid move that does not create new cookies. So, $Z \mapsto R(2, l-1)$ is the cascade we seek. End of Case 1.1

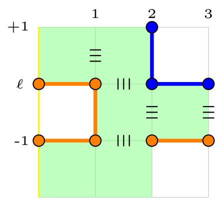

Fig. 3.8 (a). Case 1.1

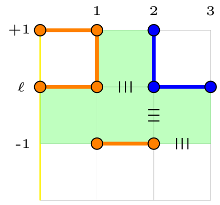

Fig. 3.8 (b). Case 1.2

CASE 1.2: $e(1; l, l+1) \in H'$. Proposition 3.1, and the assumption that $Z \in \text{ext}(H)$ imply that $R(1, l) \in \text{int}(H)$. Then, Lemma 1.14 implies that $R(0, l)$ is a small cookie of G. Note that if $e(2, 3; l-1) \in H'$, then we're back to Case 1.1, so we may assume that $e(2, 3; l-1) \notin H'$. It follows that $e(1, 2; l-1) \in H_2$ and that $R(0, l-1)$ is not a small cookie of H. Then, by Observation 3.4 and Proposition 3.3, $R(0, l) \mapsto R(1, l)$, $Z \mapsto R(1, l-1)$ is the cascade we seek. End of Case 1.2. End of Case 1.

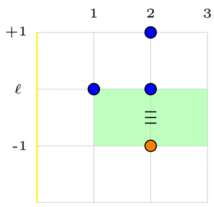

Fig. 3.9 (a). Case 2.

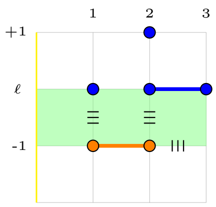

Fig. 3.9 (b). Case 2.1.

CASE 2: $v(1, l) \in H_1$. Then $e(2, 3; l) \in H_1$ or $e(2, 3; l) \notin H'$.

CASE 2.1: $e(2, 3; l) \in H_1$. Note that if $e(2, 3; l-1) \in H_2$, then we're back to essentially the same scenario as Case 1.1, so we may assume that $e(2, 3; l-1) \notin H_2$. Then $e(1, 2; l-1) \in H_2$. It follows that $e(1; l-1, l) \notin H'$ and so $R(0, l-1)$ is not a small cookie of H. Now, either $e(1, 2; l) \in H_1$, or $e(1, 2; l) \notin H'$.

CASE 2.1 (a): $e(1, 2; l) \in H_1$. By Observation 3.4 and Proposition 3.3, $Z \mapsto R(1, l-1)$ is the cascade we seek. End of Case 2.1(a).

CASE 2.1 (b): $e(1, 2; l) \notin H'$. Then we have that $e(2; l, l+1) \in H_1$, that $e(1; l, l+1) \in H_1$ and that $R(1, l) \in \text{int}(H)$. It follows that $R(0, l) \in \text{ext}(H)$, so $R(0, l)$ must be a small cookie. Then, after $R(0, l) \mapsto R(1, l)$, we are back to Case 2.1 (a). End of Case 2.1(b). End of Case 2.1.

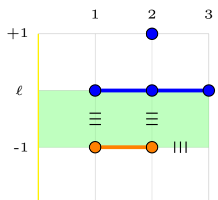

Fig. 3.10 (a). Case 2.1(a).

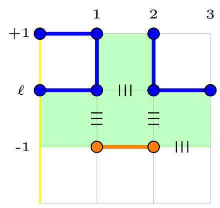

Fig. 3.10 (b). Case 2.1(b).

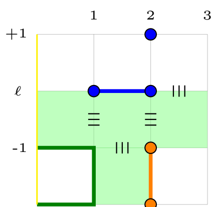

Fig. 3.11. Case 2.2.

CASE 2.2: $e(2, 3; l) \notin H'$. Then we have that $e(1, 2; l) \in H_1$. Note that if $e(1, 2; l-1) \in H_2$, then we're back to Case 2.1, so we may assume that $e(1, 2; l-1) \notin H'$. It follows that $e(2; l-2, l-1) \in H_2$. Note that if $e(1; l-1, l) \in H_1$, then $R(0, l-1) \in \text{ext(H)}$ and $R(0, l) \in \text{ext(H)}$, contradicting Lemma 1.14, so we may assume that $e(1; l-1, l) \notin H'$. Then we must have that $e(1; l-2, l-1) \in H'$. This implies that $R(0, l-2)$ is a small cookie of H. Then after $R(0, l-2) \mapsto R(1, l-2)$, we're back to Case 2.1(a). End of Case 2.2. End of Case 2. □.

Observation 3.6. Let H be a Hamiltonian cycle of an $m \times n$ grid graph G, with $m, n \geq 5$ and let J be a large cookie of G. Then $J \cap R_2 \neq \emptyset$.

Lemma 3.7. Let H be a Hamiltonian cycle of an $m \times n$ grid graph G, with $m, n \geq 5$ and assume that G has at least two large cookies J_1 and J_2. Then switching the neck N_{J_1} of J_1 splits H into two cycles H_1 and H_2 such that there is $v_1 \in H_1 \cap R_2$ and $v_2 \in H_2 \cap R_2$ with v_1 adjacent to v_2.

Proof. Orient H. Let $\{v_x, v_y\}$ be the boundary edge of the neck N_{J_1} of J_1. Define $\overrightarrow{K}_1$ and $\overrightarrow{K}_2$ to be the subtrails $\overrightarrow{K}((v_x, v_{x+1}), (v_{y-1}, v_y))$ and $\overrightarrow{K}((v_y, v_{y+1}), (v_{x-1}, v_x))$ of $\overrightarrow{K}_H$, respectively. By Lemma 1.16 (i), switching N_{J_1} gives two cycles H_1 and H_2, with $V(H_1) = V(\overrightarrow{K}_1 \setminus \{(v_x, v_y)\})$ and $V(H_2) = V(\overrightarrow{K}_2)$. By Lemma 1.13, $V(J_1) = V(\overrightarrow{K}_1)$. By Observation 3.6, $V(J_1) \cap R_2 \neq \emptyset$. Then $V(H_1) \cap R_2 \neq \emptyset$.

Since $V(J_1) = V(\overrightarrow{K}_1)$, we have that $V(J_2) \subseteq V(\overrightarrow{K}_2) = V(H_2)$. By Observation 3.6, $V(J_2) \cap R_2 \neq \emptyset$. It follows that $V(H_2) \cap R_2 \neq \emptyset$.

We have shown that $V(H_1) = R_2 \neq \emptyset$ and that $V(H_2) \cap R_2 \neq \emptyset$. It remains to check there is $v_1 \in H_1 \cap R_2$ and $v_2 \in H_2 \cap R_2$ with v_1 adjacent to v_2. $v_1 \in H_1 \cap R_2$ and $R_2 = w_1, ..., w_s$, with $v_1 = w_1$. Sweep R_2 starting at w_1. If there is $i \in \{1, ..., s-2\}$ such that $v_i \in H_1$ and $v_{i+1} \in H_2$, we are done. If there is no such i then $R_2 \cup H_1$, contradicting that $R_2 \cup H_2 \neq \emptyset$. □

Proposition 3.8. (MLC Algorithm.) Let H be a Hamiltonian cycle of an $m \times n$ grid graph G. Assume that G has more than one large cookie. Then there is a cascade of length at most two that reduces the number of large cookies of G by one.

Proof. Let J be a large cookie of G with neck N_J. Switching N_J splits H into the cycles H_1 and H_2. By Lemma 3.7 there is $v_1 \in H_1 \cap R_2$ and $v_2 \in H_2 \cap R_2$ with v_1 adjacent to v_2. By Lemma 3.5 there is a cascade of length at most two, whose last move is $N_j \mapsto N'_J$ with $N_J \neq N'_J$. By Observation 3.4, this cascade decreases the number of large cookies of G by one. □

3.2 Existence of the 1LC Algorithm

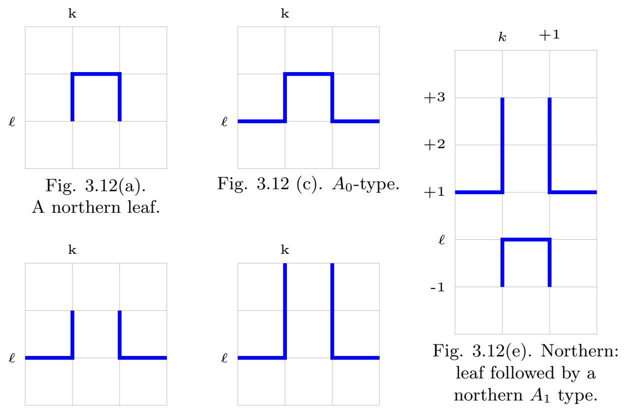

Fig. 3.12(a). A northern leaf.
Fig. 3.12 (c). A_0-type.
Fig. 3.12(e). Northern: leaf followed by a northern A_1 type.
Fig. 3.12 (b). A-type.
Fig. 3.12 (d). A_1-type.

Definitions. Let G be an $m \times n$ grid graph and H be a Hamiltonian cycle of G. We call a subpath of H on the edges $e(k; l, l+1), e(k, k+1; l+1)$ and $e(k+1; l, l+1)$ a *northern leaf.* We will often say that $R(k,l)$ is a northern leaf to mean that $e(k; l, l+1), e(k, k+1; l+1)$ and $e(k+1; l, l+1)$ belong to H. Southern, eastern and western leaves are defined analogously.

We call the subgraph of H on the edges $e(k-1, k; l), e(k; l, l+1)$, $e(k+1, k+2; l)$, and $e(k+1; l, l+1)$ a *northern A-type.* Suppose H has a northern A-type. We call the subgraph $A \cup e(k, k+1; l+1)$ of H a *northern A_0-type*, and we call the subgraph $A \cup e(k; l+1, l+2) \cup e(k+1; l+1, l+2)$ of H a *northern A_1-type.* We make analogous definitions for eastern, southern and western A-types. See Figures 3.12.

Let $R(k, l-1)$ be a northern leaf. If H has a northern A-type on $e(k-1, k; l+1), e(k; l+1, l+2)$ and $e(k+1, k+2; l+1), e(k+1; l+1, l+2)$ then we say that A-type *follows* the northern leaf $R(k, l-1)$ *northward.* We call the switchable box $R(k, l+1)$ the *switchable middle-box* of the A_1-type. Analogous definitions apply for other compass directions.
Let A be a northern A_0-type in H on the edges $e(k-1, k; 0)$, $e(k; 0, 1)$, $e(k, k+1, 1)$, $e(k+1; 0, 1)$, $e(k+1, k+2; 0)$ and let $j \in \{1, 2, .. \lfloor \frac{n}{2} \rfloor\}$. We define a *northern j-stack of A_0's starting at A* to be a subgraph *stack*$(j; A_0)$ of H, where $\text{stack}(j; A_0) = \bigcup_{i=0}^{j-1} \Big(A + (0, 2i)\Big)$. If $j = \lfloor \frac{n}{2} \rfloor$, we call the j-stack a *full j-stack of A_0's.* We note that j is the number of A_0's in stack$(j; A_0)$. Eastern, southern, and western j-stacks are defined analogously.

We denote the set of northern and southern small cookies by *SmallCookies*$\{N, S\}$ and the set of eastern and western small cookies by *SmallCookies*$\{E, W\}$. Assume that $C \in$ *SmallCookies*$\{N, S\}$ is an easternmost or westernmost southern or northern small cookie. Then we call C and *outermost* small cookie in *SmallCookies*$\{N, S\}$. Outermost small cookies in *SmallCookies*$\{E, W\}$ are defined analogously.

Let $R(k,l)$ be a northern leaf. We say that the cascade $\mu_1, ..., \mu_r$ *collects* $R(k,l)$, if μ_r is the move $Z \mapsto R(k,l)$. Note that, since $R(k,l)$ is not switchable, Z must be a switchable box adjacent to $R(k,l)$.

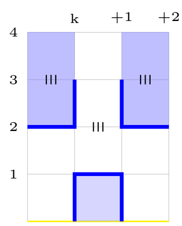

Fig. 3.13.

Given a small cookie C, we want to show that there is a cascade that collects C. For definiteness, assume that C is the northern small cookie $R(k, 0)$. If $e(k, k+1; 2) \in H$, then $C + (0,1) \mapsto C$ is the cascade we seek, so we only need to consider the case where $e(k, k+1; 2) \notin H$. Then we must have $e(k-1, k; 2) \in H$, $e(k+1, k+2; 2) \in H$, $e(k; 2, 3) \in H$ and $e(k+1; 2, 3) \in H$. Note that if $e(k-1, k; 3) \in H$, then $C + (0, 2) \mapsto C + (-1, 2)$ followed by $C + (0, 1) \mapsto C$ is the cascade we seek, so we consider the case where $e(k-1, k; 3) \notin H$ and, by symmetry, where $e(k+1, k+2; 3) \notin H$. Now, we either have $e(k; 3, 4) \in H$ and $e(k+1; 3, 4) \in H$ or $e(k, k+1; 3) \in H$. That is, C is followed northward by an A_0-type or by an A_1-type. See Figure 3.13. From this point onward, we will omit the compass direction when it does not introduce ambiguity. We coalesce this paragraph into the following lemma:

Proposition 3.10. (1LC Algorithm). Let G be an $m \times n$ grid graph, let H be a Hamiltonian cycle of G. If H has exactly one large cookie and at least one small cookie, then there is a cascade of length at most $\frac{1}{2}\max(m,n)+\min(m,n)+2$ moves that reduces the number of small cookies of H by one and such that it does not increase the number of large cookies.

The proof of Proposition 3.10 requires the following two Lemmas.

Lemma 3.11. Let G be an $m\times n$ grid graph, let H be a Hamiltonian cycle of G, and let $C\in SmallCookies\{N,S\}$ be an easternmost small cookie. Assume that G has only one large cookie. Then there cannot be a full j-stack of $A_0's$ starting at the A_0-type that contains C.

Lemma 3.12. Let G be an $m\times n$ grid graph, let H be a Hamiltonian cycle of G, and let $C\in SmallCookies\{N,S\}$ be an easternmost small cookie. Assume that there is a j-stack of A_0 starting at the A_0-type containing C. Let L be the leaf in the top (j^{th}) A_0 of the stack. Assume that L is followed by an A_1-type and that G has only one large cookie. Then there is a cascade of at most $\min(m,n)+3$ moves that collects L.

Proof of Proposition 3.10. Since there is at least one small cookie, at least one of $SmallCookies\{N,S\}$ and $SmallCookies\{E,W\}$ is nonempty. WLOG assume that $SmallCookies\{N,S\}$ is nonempty. Let $C\in SmallCookies\{N,S\}$ be an easternmost small cookie. We will require the following two lemmas.

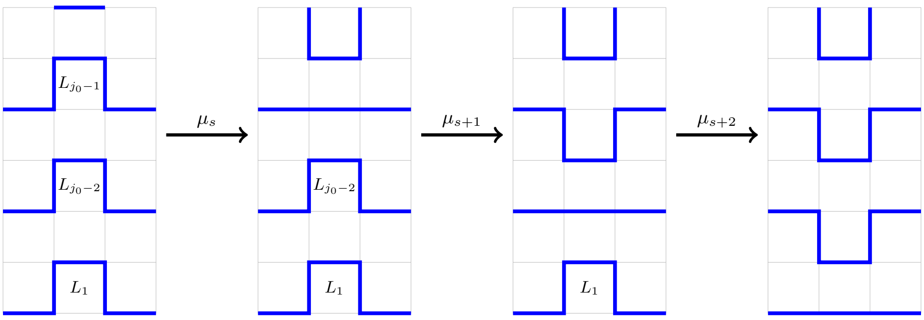

Fig. 3.14 (a). Fig. 3.14 (a). Fig. 3.14 (c). Fig. 3.14 (d).

For definiteness assume that C is a small northern cookie on the southern boundary. Let $Q(j)$ be the statement "There is a j-stack of A_0's starting at the A_0-type containing C". Note that C is contained in an A_0-type, so $Q(1)$ is true. By Lemma 3.11, there is a $j_0\in\left\{2,3,...\left\lfloor\frac{n}{2}\right\rfloor\right\}$ such that for each $j\in\{2..,j_0-1\}$, $Q(j)$ is true for each $j<j_0$ but $Q(j_0)$ is not true.
For $j\in\{1..,j_0-1\}$ let L_j be the northern leaf of the j^{th} A_0-type in the stack. Note that $L_1=C$. Lemma 3.9 implies that L_{j_0-1} is either followed by an A_1-type or there is a cascade that collects L_{j_0-1}. If L_{j_0-1} is followed by an A_1-type, then by Lemma 3.12, we can find a cascade that collects it, so we only need to check the case in which there is a cascade $\mu_1,...,\mu_s$ that collects L_{j_0-1}. Note that μ_s must be the move $L_{j_0-1}+(1,0)\mapsto L_{j_0-1}$. Then $\mu_1,...,\mu_s,L_{j_0-2}+(0,1)\mapsto L_{j_0-2},...,L_1+(0,1)\mapsto L_1$ is a cascade that collects C. See Figures 3.14 for an illustration with $j_0-1=3$. Note that $j\leq\frac{n}{2}$, and that by Lemma 3.12, there are at most $\min(m,n)+3$ moves required to collect L. After that, we need at most another $j-1$ flips to collect C, so C can be collected after at most $\frac{1}{2}\max(m,n)+\min(m,n)+2$ moves. See Figure 3.14 for an illustration with $j_0-1=3$. □

It remains to prove Lemmas 3.11 and 3.12.

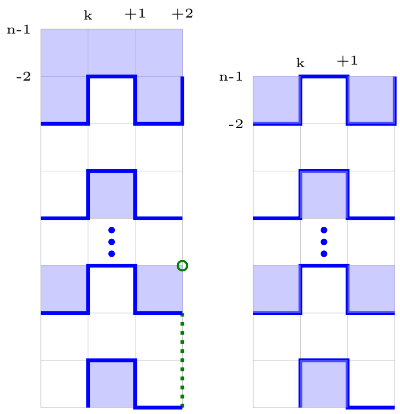

Fig 3.15(a). Case 1.1. Fig 3.15(b). Case 1.2.

Proof of Lemma 3.11. Assume for contradiction that C is in a full stack of A_0's starting at the A_0 that contains C. For definiteness, assume that $C=R(k,0)$ is a small northern cookie on the southern boundary. First we check that $m-1>k+2$. If $m-1=k+2$, then we must have $e(k+2;0,1)\in H$ and $e(k+2;1,2)\in H$. But then H misses $v(k+2,3)$ (in green in Figure 3.15 (a)). The number j of A_0's in the full stack is even or an odd so there are two cases to check. Note that for each odd $i\in\{1,2,...,j\}$, the leaf of the i^{th} A_0 belongs to $\text{ext}(H)$ and for each even $i\in\{1,2,...,j\}$, the leaf of the i^{th} A_0 belongs to $\text{int}(H)$.

CASE 1: j is even. Note that the top leaf of the stack is in $\text{int}(H)$. Now, $n-1$ is either even or odd.

CASE 1.1: $n-1$ is even. We have that $R(k,n-3)\in\text{int}(H)$. But then we must have $R(k,n-2)\in\text{ext}(H)$ and $R(k+1,n-2)\in\text{ext}(H)$, contradicting Lemma 1.14. End of Case 1.1. See Figure 3.15(a).

CASE 1.2: $n-1$ is odd. Then we must have that $R(k+1, n-2)$ is a small southern cookie. But this contradicts our assumption that C is the easternmost small cookie in $SmallCookies\{N, S\}$. End of Case 1.2. End of Case 1. See Figure 1.15 (b).

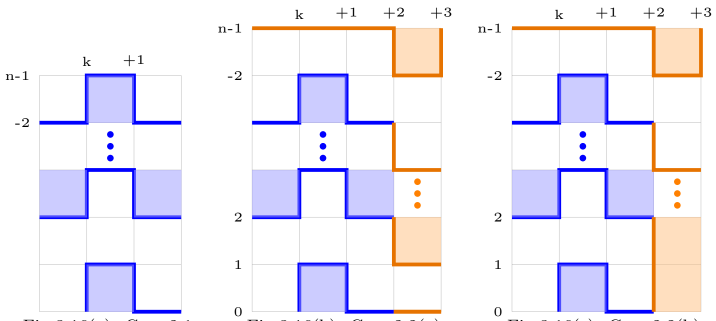

Fig 3.16(a). Case 2.1. Fig 3.16(b). Case 2.2(a). Fig 3.16(c). Case 2.2(b).

CASE 2: j is odd. Note that the top leaf of the stack is in ext(H). Again, $n-1$ is either even or odd.

CASE 2.1: $n-1$ is odd. We have that $R(k, n-2) \in$ ext(H). But then, the fact that $e(k, k+1; n-1) \in H$ implies that $R(k, n-2)$ is not a cookie neck, contradicting Lemma 1.14. End of Case 2.1. See Figure 3.16 (a).

CASE 2.2: $n-1$ is even. We have that $e(k+1, k+2; 0) \in H$. Then, either $e(k+2, k+3; 0) \in H$, or $e(k+2; 0, 1) \in H$.

CASE 2.2(a): $e(k+2, k+3; 0) \in H$. Then we must have $e(k+2; 1, 2) \in H$. Note that for $i \in \{1, 3, ..., n-2\}$, $e(k+2; i, i+1) \in H$ implies $e(k+2; i+2, i+3) \in H$. Then, for $i \in \{1, 3, ..., n-2\}$, we have that $e(k+2; i, i+1) \in H$. Note that we must also have $e(k+2, k+3; n-2) \in H$. Then $R(k+2, n-2)$ must be a southern small cookie, contradicting the easternmost assumption. End of Case 2.2 (a).

CASE 2.2(b): $e(k+2; 0, 1) \in H$. Note that if $e(k+2, k+3; 1) \in H$, then $R(k+2, 0)$ must be a small cookie, contradicting the easternmost assumption. Then $e(k+2, k+3; 1) \notin H$. But then we have $e(k+2; 1, 2) \in H$, and we are back to Case 2.2(a). End of Case 2.2(b). End of Case 2.2. End of Case 2. $\square$

We will need Lemmas 3.13-3.16 to prove Lemma 3.12.

Lemma 3.13. Let G be an $m \times n$ grid graph, and let H be a Hamiltonian cycle of G. Let C be a small cookie of G. Assume that G has only one large cookie, and that there is a j-stack of A_0 starting at the A_0-type containing C. Let L be the leaf in the top (j^{th}) A_0 of the stack, and assume that L is followed by an A_1-type. Let X and Y be the boxes adjacent to the middle-box of the A_1-type that are not its H-neighbours. If $P(X, Y)$ has no switchable boxes, then either:

(i) there is a cascade of length at most $\min(m, n)$, which avoids the stack of A_0's, and after which $P(X, Y)$, gains a switchable box, or

(ii) there is a cascade of length at most $\min(m, n) + 1$, that collects L and avoids the stack of A_0's.

We postpone the proof of Lemma 3.13 until Section 4. It takes up all of the section.

Lemma 3.14. Let G be an $m \times n$ grid graph, let H be a Hamiltonian cycle of G, and let $C \in SmallCookies\{N, S\}$ be an easternmost small cookie. Assume that G has only one large cookie, and that there is a j-stack of A_0 starting at the A_0-type containing C. Let L be the leaf contained in the top (j^{th}) A_0 of the stack. Assume that L is followed by an A_1-type with looping H-path $P(X, Y)$. Let X' be the box of G that shares edges with X and Y. Then $X' \in G_2$.

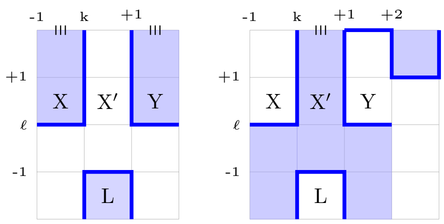

Fig. 3.17 (a). Case 1. Fig. 3.17 (b). Case 2.

Proof. For definiteness, assume that L is the northern leaf $R(k, l-2)$, and that $X = R(k-1, l)$. Then $X' = R(k, l)$ and $Y = R(k+1, l)$. Note that $l-2 \geq 0$ and $l+2 \leq n-1$. Either $P(X, Y)$ is contained in ext(H), or $P(X, Y)$ is contained in int(H), so there are two cases to check.

CASE 1: $P(X, Y) \subset ext(H)$. By Lemma 1.14, we must have that $m-1 > k+2$ and $k-1 > 0$. To see that $n-1 > l+2$, assume for contradiction that $n-1 = l+2$. By Lemma 1.14, $X + (1, 0)$ and $Y + (1, 0)$ are cookie necks. But this contradicts the assumption that there is only large cookie in G. See Figure 3.17 (a). End of Case 1.

CASE 2: $P(X,Y) \subset int(H)$. By Lemma 1.14, we must have that $m-1 > k+2$ and $k-1 > 0$. To see that $n-1 > l+2$, assume for contradiction that $n-1 = l+2$. Lemma 1.14 implies that $X' + (0,1)$ is the neck of the large cookie of G. But now $X' + (2,1)$ must be a small cookie of G, contradicting the easternmost assumption. See Figure 3.17 (b).

Lemma 3.15. Let G be an $m \times n$ grid graph, let H be a Hamiltonian cycle of G, and let $C \in SmallCookies\{N,S\}$ be an easternmost small cookie. Assume that G has only one large cookie, and that there is a j-stack of A_0 starting at the A_0-type containing C. Let L be the leaf in the top (j^{th}) A_0 of the stack. Assume that $L \in \text{int}(H)$ and that L is followed by an A_1-type with looping H-path $P(X,Y)$. Let X' be the box of G that shares edges with X and Y. If X' is not in G_3 then, either $X' \mapsto W$ is a cascade, or there is a cascade $\mu, X' \mapsto W$, of length two, with $X' \mapsto W$ nontrivial in either case.

Proof. Suppose that X' is not in G_3. By Lemma 3.14, $X' \in G_2 \setminus G_3$. For definiteness assume that L is a northern leaf, and let $X' = R(k,l)$. The assumption that $L \in \text{int}(H)$ implies that $l-2 > 0$.

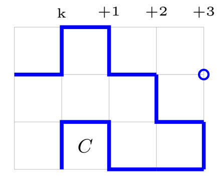

Fig 3.17$\frac{1}{2}$. $m-1 = k+3$

Now we check that $m-1 > k+3$. By Lemma 1.14, $m-1 > k+2$. Assume for contradiction that $m-1 = k+3$. Note that we must have $e(k+1,k+2;0) \in H$, $e(k+2,k+3;0) \in H$, and $e(k+3;0,1) \in H$. This implies that we must have $e(k+2;1,2) \in H$ and $e(k+2,k+3;1) \in H$. But now H misses $v(k+3;2)$. It follows that we must have $m-1 > k+3$. See Figure 3.17$\frac{1}{2}$(b). By symmetry, $0 < k-2$. It follows that $l+3 = n-1$.

The same argument used in Case 2.2 of Lemma 3.11 (see Figures 3.16 (b) and (c)) shows that we have $e(k+2;l+1,l+2) \in H$ and $e(k+2,k+3;l+1) \in H$. Now either $e(k,k+1;l+2) \in H$ or $e(k,k+1;l+2) \notin H$. See figures 3.18 (a) and (b).

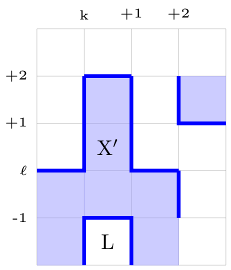

Fig. 3.18 (a). Case 1.

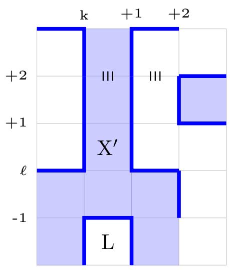

Fig. 3.18 (b). Case 2.

CASE 1: $e(k,k+1;l+2) \in H$. By Lemma 1.16, $X' \mapsto X' + (1,1)$ is a valid move, and by Observation 3.4, $X' \mapsto X' + (1,1)$ creates no new cookies. End of Case 1.

CASE 2: $e(k,k+1;l+2) \notin H$. Lemma 1.14 implies that $e(k+1,k+2;l+2)$ cannot be in H either. It follows that $X' + (0,2)$ must be the neck of the large cookie. The assumption that there is only one large cookie implies that $e(k+2;l+2,l+3) \notin H$. Then we must have $e(k+2,k+3;l+2) \in H$. Then, by Lemma 1.16, $X' + (1,1) \mapsto X' + (2,1)$, $X' \mapsto X' + (1,0)$ is the cascade we seek. □

Lemma 3.16. Let H be a Hamiltonian cycle of an $m \times n$ grid graph G. Let $X' \in G_3 \cap \text{ext}(H)$ be a switchable box, and let $P(X,Y)$ be the looping H-path of X'. Assume that $P(X,Y)$ has a switchable box in $G_0 \setminus G_2$. Then switching X' splits H into two cycles H_1 and H_2 such that there is $v_1 \in H_1 \cap R_2$ and $v_2 \in H_2 \cap R_2$ with v_1 adjacent to v_2.

Proof. Let Z be a switchable box of $P(X,Y)$ in $G_0 \setminus G_2$. Orient H. Let (v_x, v_{x+1}) and (v_{y-1}, v_y) be the edges of X' in H. Define $\overrightarrow{K}_1$ and $\overrightarrow{K}_2$ to be the subtrails $\overrightarrow{K}((v_x, v_{x+1}), (v_{y-1}, v_y))$ and $\overrightarrow{K}((v_y, v_{y+1}), (v_{x-1}, v_x))$ of $\overrightarrow{K}_H$, respectively. By Lemma 1.16 (i), switching X' gives two cycles H_1 and H_2, with $V(H_1) = V(\overrightarrow{K}_1 \setminus \{v_x, v_y\})$ and $V(H_2) = V(\overrightarrow{K}_2)$. By Proposition 3.1, Z has a vertex in H_1 and another in H_2, and the same holds for X'. Since $X' \in G_3$ and $Z \in G_0 \setminus G_2$, by JCT, $H_1 \cap R_2 \neq \emptyset$. Similarly, $H_2 \cap R_2 \neq \emptyset$. Now the argument in the last paragraph of Lemma 3.7 shows that there must be $v_1 \in H_1 \cap R_2$ and $v_2 \in H_2 \cap R_2$ with v_1 adjacent to v_2. □

Proof of Lemma 3.12. For definiteness assume that L is the northern leaf $R(k,l)$. Let $P(X,Y)$ be the looping H-path following L, with $X = R(k-1,l+2)$ and $Y = R(k+1,l+2)$. By Lemma 3.13, either there is a cascade that collects L, or a cascade after which $P(X,Y)$ gains a switchable box, with both cascades having length at most $\min(m,n)+1$, and both avoiding the j-stack of A_0's starting at C. If the former, we are done, so may assume that $P(X,Y)$ has a switchable box Z. Let J be the large cookie of G and let N_J be the neck of J. Note that N_J is not a box of $P(X,Y)$. Now, $P(X,Y)$ is either contained in ext(H) or int(H).

CASE 1: $P(X,Y) \subseteq \text{ext(H)}$. Then $X' \subset \text{int(H)}$. By Lemma 3.14, $X' \in G_2$. By Proposition 3.3, $Z \mapsto X'$ is a valid move. By Observation 3.4, $Z \mapsto X'$ does not create additional cookies. Then $Z \mapsto X'$,

$L + (0,1) \mapsto L$ is a cascade that collects L.

CASE 2: $P(X,Y) \subseteq \text{int}(H)$ .Then $X' \subset \text{ext(H)}$. If $Z \subset G_2$, by Proposition 3.3 $Z \mapsto X'$ is a valid move, and by Observation 3.4, $Z \mapsto X'$ does not create additional cookies. Then $Z \mapsto X'$, $L \mapsto L + (0,1)$ is a cascade that collects L.

Suppose then that $Z \subset G_0 \setminus G_2$. By Lemma 3.15, we only need to check the case where $X' \in G_3$. Note that switching X' splits H into two cycles H_1 and H_2. By Lemma 3.16 there is $v_1 \in H_1 \cap R_2$ and $v_2 \in H_2 \cap R_2$ with v_1 adjacent to v_2. By Lemma 3.5 there is a cascade $X' \mapsto W$, or $\mu, X' \mapsto W$, with $X' \mapsto W$ nontrivial. Note that here, X' plays the role that Z played in Lemma 3.5. Then $\mu, X' \mapsto W, L + (0,1) \mapsto L$ or $X' \mapsto W, L + (0,1) \mapsto L$ is a cascade that collects L.

We have just shown that if $P(X,Y)$ has a switchable box, then the cascade required to collect L has length at most three. By Lemma 3.13, the cascade after which $P(X,Y)$ to gains a switchable box has length at most $\min(m,n)$. Thus, at most $\min(m,n) + 3$ moves are required to collect L. □

3.3 Summary

• In Section 3 we proved the MLC and 1LC algorithms. The proof of the MLC algorithm is fully contained here, while the proof of the 1LC algorithm depends on Lemma 3.13, whose proof is given in Section 4.

Proposition 3.3 characterizes when double-switch moves are valid and serves as the primary tool for both algorithms.

The MLC algorithm handles the case where H has multiple large cookies. To collect a large cookie J with switchable neck N_J, we look for a switchable box Z in the looping H-path of N_J. Proposition 3.8 shows that either such a Z already exists, or a there is single preparatory move that produces one.

The 1LC algorithm handles the case where H has exactly one large cookie and at least one small cookie. It collects outermost small cookies. Suppose that C is an outermost small cookie. Either C can be collected immediately by a single move, or C is followed by a j-stack of A_0-types and an A_1-type with switchable middle-box X'. If the latter, let $P(X,Y)$ be the H-path determined by X'. If $P(X,Y)$ contains a switchable box Z, then C can be collected by either switching X' directly (if $Z \in G_2$) or by using Lemma 3.5 to find a cascade of length at most two that enables switching X (if $Z \in G_0 \setminus G_2$). In both cases, a cascade of flips then collects C. The existence of such a switchable box Z is guaranteed by Lemma 3.13, whose proof takes up Section 4. •

4 Looping fat paths, turns and weakenings

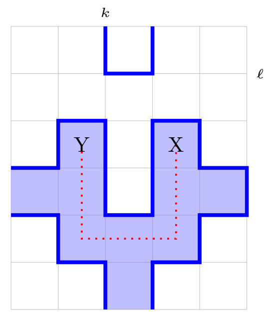

Fig. 4.1. A southern looping fat path $G\langle N[P(X,Y]\rangle$. $N[P(X,Y]$ shaded in light blue. $P(X,Y)$ traced in red.

Definitions. Let $J = \{X_1, X_2, ..., X_r\}$ be a collection of boxes in an $m \times n$ grid graph G, and let H be a Hamiltonian cycle of G. We will use the notation $G\langle J\rangle$ to denote the subgraph of G with vertex set $V(G\langle J\rangle) = V(J)$ and edge set $E(G\langle J\rangle) = E(J) \cap E(H)$. The boxes of $G\langle J\rangle$ are the boxes of J. We call $G\langle J\rangle$ the *subgraph of G induced by J*.

Suppose that the southern leaf $R(k,l)$ is followed by an A_1-type. Let $X = R(k+1, l-2)$ and $Y = R(k-1, l-2)$. Let $P(X,Y)$ be a southern looping H-path following the southern leaf $R(k,l)$. The set of all boxes in $P(X,Y)$, along with their H-neighbours, is called the *H-neighbourhood of* $P(X,Y)$, and is denoted by $N[P(X,Y)]$. Consider the subgraph $F = G\langle N[P(X,Y)]\rangle$ of G induced by $N[P(X,Y)]$. We define a *short weakening of F* to be a cascade of length three or less after which, the edge $\{v(k, l-1), v(k+1, l-1)\}$ is in the resulting Hamiltonian cycle of G. We say that F is a *southern looping fat path* if F has no short weakening. We define western, northern and eastern looping fat paths analogously.

Assume that G has only one large cookie, and that there is a j-stack of A_0 starting at the A_0-type containing an outermost southern small cookie C. Let L be the leaf in the top (j^{th}) A_0 of the stack, and assume that L is followed by an A_1-type with looping H path $P(X,Y)$. Let $F = G\langle N[P(X,Y)]\rangle$. If F has no short weakening, we say that F a southern looping fat path *anchored* at the outermost small southern cookie C. Analogous definitions apply for northern, eastern, and western looping fat paths.

We remark that if $P(X,Y)$ has a switchable box then, by Proposition 3.3 and the proof of Lemma 3.12, F has a short weakening, and thus it cannot be a looping fat path.

Throughout the remainder of this section, we assume that G has exactly one large cookie, and that all looping fat paths considered are anchored at some outermost small cookie.

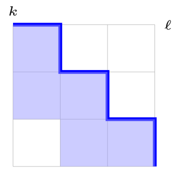

Fig. 4.2. $S_{\rightarrow}(k,l;$ $k+3, l-3)$.

We define below a subgraph of G consisting of the union of translations of two adjacent and perpendicular edges of G. Let $r \in \mathbb{N}$, and let *the stairs from (k,l) to (k+r,l-r) east* be denoted by $S_{\rightarrow}(k,l;k+r,l-r)$ and be defined as:

$$S_{\rightarrow}(k,l;k+r,l-r) = \bigcup_{j=0}^{r-1} \Big(e(k,k+1;l) + (j,-j)\Big) \cup \Big(e(k+1;l-1,l) + (j,-j)\Big).$$

We define $d(S) = r$ to be the *length* of $S_{\rightarrow}(k,l;k+r,l-r)$. We say that $S_{\rightarrow}(k,l;k+r,l-r)$ starts at $v(k,l)$ and ends at $v(k+r, l-r)$. The subscripted arrow indicates the direction from $v(k,l)$ of the first edge of the subgraph. By choosing an "up", "down", "left" or "right" arrow for direction and a sign for the third and fourth arguments of $S_{\square}(k,l;k \pm r, l \pm r)$ we may describe any of the eight possible steps subgraphs starting at the vertex $v(k,l)$. See Figure 4.2.

Definitions. • Let H be a Hamiltonian cycle of an $m \times n$ grid graph G. Let T be the subgraph of H on the edges $S_{\downarrow}(k+1,l;k',l'+1)$, $e(k;l-1,l)$ and $e(k'-1,k';l')$, where $k' = k + d(T)$, $l' = l - d(T)$, where $d(T) = d(S) + 1$ is *the length of T* and $d(T) \geq 2$. We call T a *north-east turn*. If both $e(k,k+1;l)$ and $e(k';l',l'+1)$ belong to $G \setminus H$, call T an *open north-east turn*. If exactly one of $e(k,k+1;l)$ and $e(k';l',l'+1)$ is in H, then T is a *half-open north-east turn*. If both $e(k,k+1;l) \in H$ and $e(k';l',l'+1) \in H$, then T is a *closed north-east turn*. See Figure 4.3 For any north-east turn T, we say that $R(k, l-1)$ is the *northern leaf* of T and $R(k'-1, l')$ is the *eastern leaf* of T. If $e(k,k+1;l) \notin H$ we call $R(k,l-1)$ an *open northern leaf of T* and if $e(k,k+1;l) \in H$ we call $R(k,l-1)$ a *closed northern leaf of T*. We note that the two leaves of a turn will determine its "leaf prefix": If a turn has a north leaf and an east leaf then the turn is a north-east turn.

We say that a looping fat path F *has a turn* (open, half-open or closed) to mean that there exists some turn T of H such that $E(F) \supset E(T)$.

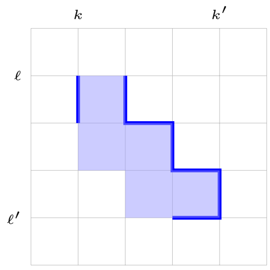

Fig. 4.3. A half-open northeast turn.

Sketch of proof of Lemma 3.13. Let H be a Hamiltonian cycle of an $m \times n$ grid graph G. Assume that $P(X,Y)$ is a looping H-path with no switchable boxes, following a leaf L. It follows that $P(X,Y)$ is contained in a looping fat path F. In Section 4.2, we show that every looping fat path must have a turn. In Sections 4.3 we show that given a turn, we can find a cascade we call a weakening (precise definition in Section 4.3) that collects at least one of the leaves of the turn. Then we show that after such a cascade, either $P(X, Y)$ gains a switchable box, or we can extend the cascade by a single move to collect L. The rest of the Section is organized as follows. Section 4.1 proves structural properties of fat paths, on which the later sections build. Section 4.2 shows that every fat path contains a turn (Proposition 4.7). In Section 4.3 we define weakenings, prove that turns have weakenings, and give a proof of Lemma 3.13. •

4.1 Properties of looping fat paths

Lemma 4.1. Let $F = G\langle N[P(X,Y)]\rangle$ be a looping fat path. Let $W = R(k,l)$ [2] be a box of $P = P(X,Y)$ with the H-neighbour $Z = W + (0,-1)$ southward in $N[P] \setminus P$. Then:

(a) Z has exactly one H-neighbour in P and W has no other H-neighbour in $N[P] \setminus P$.

(b) If W is not an end-box (i.e. X or Y) of P, then the H-neighbours of W in P are $W + (-1,0)$ and $W + (1,0)$. Furthermore, $S_{\rightarrow}(k-1,l;k,l-1) \in H$, $S_{\uparrow}(k+1,l-1;k+2,l) \in H$, and $(k,k+1;l+1) \in H$.

(c) If W is an end-box of P, then $e(k,k+1;l+1) \in H$ and exactly one of $e(k;l,l+1)$ and $e(k+1;l,l+1)$ belong to H.

(d) Z is a leaf or Z is a switchable box in H.

Analogous statements apply when Z is west, north or east of W.

Proof of (a). Note that if Z has more than one H-neighbour in P then we can make an H-cycle. To see that W has no other H-neighbour in $N[P] \setminus P$, assume for contradiction that W has at least two H-neighbours in $N[P] \setminus P$.

If W is an end-box of P, then, by definition of A_1, W has at most two H-neighbours, and at least one of them must belong to P, contradicting our assumption that W has at least two H-neighbours in $N[P] \setminus P$.

If W is not an end-box, then W must have four H neighbours: two in $N[P] \setminus P$ and two in P. By definition of a looping fat path, at least one of the neighbours of W in P, say W', is not an end-box. But then W' must be switchable, contradicting that F is a looping fat path. □

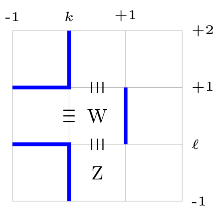

Fig. 4.4 (a).

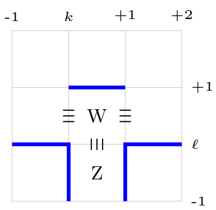

Fig. 4.4 (b).

Proof of (b). First we show that the H-neighbours of W are $W + (1,0)$ and $W + (-1,0)$. Assume for contradiction that $W + (1,0)$ is not an H-neighbour of W. Then the H-neighbours of W in P must be $W + (-1,0)$ and $W + (0,1)$. It follows that $S_{\rightarrow}(k-1,l;k,l-1) \in H$ and $S_{\rightarrow}(k-1,l+1;k,l+2) \in H$. Note that, by the definition of A_1 and looping fat paths, $W + (-1,0)$ is not an end-box of P. But then $W + (-1,0)$ is a switchable box of P, contradicting that F is a looping fat path. Therefore, the H-neighbours of W in P are $W + (-1,0)$ and $W + (1,0)$. It follows that $S_{\rightarrow}(k-1,l;k,l-1) \in H$ and $S_{\uparrow}(k+1,l-1;k+2,l) \in H$, and by part (a), $(k,k+1;l+1) \in H$. See Figure 4.4 (a) and (b). End of proof for (b).

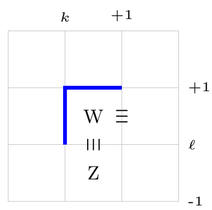

Fig. 4.5 (a).

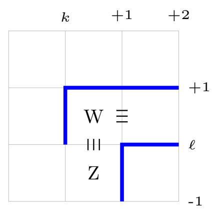

Fig. 4.5 (b). Case 1: F is eastern.

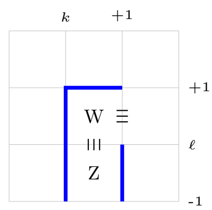

Fig. 4.5 (c). Case 1: F is southern.

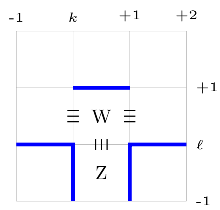

Fig. 4.5 (d). Case 2.

Proof of (c). By part (a) and the assumption that W is an end-box of P, W has exactly one H-neighbour in P and no other H-neighbours in $N[P] \setminus P$. It follows that W has exactly two edges in H and two edges not in H. Assume for contradiction that the other edge of W not in H is $e(k,k+1;l+1)$. But then $e(k;l-1,l) \in H$ and $e(k+1;l-1,l) \in H$ and the W is switchable, contradicting that F is a looping fat path. It follows that $e(k,k+1;l+1) \in H$. Since W has exactly two edges in H, we have that exactly one of $e(k;l,l+1)$ and $e(k+1;l,l+1)$ belong to H. See Figure 4.5 (a). End of proof for (c).

Proof of (d). W is either an end-box of P or it is not.

[2] $W = R(k,l)$ is not related to the southern leaf $R(k,l)$ in the definitions in page 18.

CASE 1: W is an end-box of P. By part (c), we may assume WLOG that $e(k; l, l+1) \in H$ and $e(k+1; l, l+1) \notin H$. Then F is eastern or southern. Suppose that F is eastern. Then $e(k+1, k+2; l+1) \in H$. It follows that $S_{\uparrow}(k+1, l-1; k+2, l) \in H$. But then $W + (1, 0) \in P$ is switchable, contradicting that F is a looping fat path. So F must be southern. Then $e(k; l-1, l) \in H$. It follows that $S_{\uparrow}(k+1, l-1; k+2, l) \in H$. Now, either $e(k, k+1; l-1) \in H$, or $e(k, k+1; l-1) \notin H$. Either way, (d) is satisfied. See figures 4.5 (b) and (c). End of Case 1.

CASE 2: W is not end-box of P. By part (b), the H-neighbours of W in P are $W + (-1, 0)$ and $W + (1, 0)$ and we have that $S_{\rightarrow}(k-1, l; k, l-1) \in H$, $S_{\uparrow}(k+1, l-1; k+2, l) \in H$. Then, either $e(k, k+1; l-1) \in H$ or $e(k, k+1; l-1) \notin H$. Either way, (d) is satisfied. See Figure 4.5(d). $\square$

Definitions. Let G be an $m \times n$ grid graph, let H be a Hamiltonian cycle of G and let J be an H-subtree of an H-component of G. We say that a box Z of J is a *border box* of J if Z is an H-neighbour of a box $Z' \in G_{-1} \setminus J$. We will call the edge that Z and Z' share a *shadow edge* of J. We call the set of all shadow edges of J the *shadow border* of J and denote it by $hb(J)$. We define the *shadow of* J to be the graph $h(J)$ with vertex set $V(h(J)) = V(J)$ and edge set $E(h(J)) = (E(J) \cap E(H)) \cup hb(J)$. The boxes of the shadow of J are the same as the boxes of J. We note that shadow edges cannot be incident on boxes of P.

Observation 4.2. Let $F = G\langle N[P(X, Y)]\rangle$ be a looping fat path. Then the shadow edges of the H-subtree $N[P]$ can only be incident on boxes of $N[P] \setminus P$. Moreover, exactly one of the two boxes incident on a shadow edge of $N[P]$ belongs to $N[P] \setminus P$, and the other belongs to $G_{-1} \setminus N[P]$.

Lemma 4.3. Let G be an $m \times n$ grid graph, let H be a Hamiltonian cycle of G and let $F = G\langle N[P(X, Y)]\rangle$ be a looping fat path in G. Then $E(h(F))$ is a Hamiltonian cycle of $h(F)$.

Proof. Since every vertex of $h(F)$ is incident on some edge of $E(h(F))$, it is sufficient to show that $E(h(F))$ is a cycle. We will prove:
(i) Every vertex of $h(F)$ has degree two in $h(F)$.
(ii) $E(h(F))$ is connected.
Proof of (i). Let $v = v(k, l) \in V(F)$ and let $R(k-1, l-1) = Z$. Either v has a shadow edge incident on it or it does not.

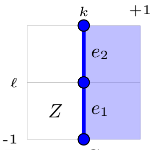

Fig. 4.6. Case 1.1.

CASE 1. v has no shadow edge incident on it. Then there are two edges e_1 and e_2 of H incident on v. These edges are either colinear or not colinear, so there are two cases to check.

CASE 1.1. e_1 and e_2 are colinear. For definiteness assume that $e_1 = e(k; l-1, l)$, $e_2 = e(k; l, l+1)$ and that $Z + (1, 0) \in F$. Since $e(k, k+1; l)$ is not a shadow edge, $Z + (1, 1) \in F$ as well, $v(k; l-1) \in F$ and $v(k; l+1) \in F$. Since $e(k-1, k; l)$ is not a shadow edge, we have that $\deg_{h(F)}(v) = 2$. See Figure 4.6. End of Case 1.1.

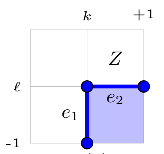

Fig. 4.7 (a). Case 1.2. $Z + (0, -1) \in F$.

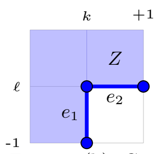

Fig. 4.7 (b). Case 1.2. $Z + (0, -1) \notin F$.

CASE 1.2. e_1 and e_2 are not colinear. For definiteness assume that $e_1 = e(k; l-1, l)$, $e_2 = e(k, k+1; l)$. Then $Z + (1, 0) \in F$ or $Z + (1, 0) \notin F$.

If $Z + (1, 0) \in F$, then $v(k; l-1) \in F$ and $v(k+1; l) \in F$. As v has no shadow edges incident on it, it follows that $\deg_{(h(F))}(v) = 2$. See Figure 4.7 (a).

If $Z + (1, 0) \notin F$, then at least one of Z, $Z + (0, 1)$ and $Z + (1, 1)$ belong to F. If $Z \in F$, since $e(k-1, k; l)$, is not a shadow edge, we have that $Z + (0, 1) \in F$. Similarly, $Z + (1, 1) \in F$. Then $v(k; l-1) \in F$ and $v(k+1; l) \in F$. As v has no shadow edges incident on it, it follows that $\deg_{(h(F))}(v) = 2$. The cases where $Z + (0, 1) \in F$ and $Z + (1, 1) \in F$ are similar and we omit them. See Figure 4.7(b). End of Case 1.2.

CASE 2. v has a shadow edge incident on it. For definiteness, let $e(k; l-1, l)$ be the shadow edge on which v is incident. There are three possibilities: $e(k-1, k; l) \in H$ and $e(k; l, l+1) \in H$, $e(k-1, k; l) \in H$ and $e(k, k+1; l) \in H$, and, $e(k; l, l+1) \in H$ and $e(k, k+1; l) \in H$. By symmetry, we only need to check the first two.

CASE 2.1: $e(k-1, k; l) \in H$ and $e(k; l, l+1) \in H$. Then $e(k, k+1; l) \notin H$. By Observation 4.2, exactly one of Z and $Z + (1, 0)$ belongs to $N[P(X, Y)] \setminus P(X, Y) = N[P] \setminus P$ and the other belongs to $G_{-1} \setminus N[P]$.

Note that by Lemma 4.1 (d), $Z+(1,0) \notin N[P] \setminus P$. Then we must have $Z+(1,0) \in G_{-1} \setminus N[P]$ and $Z \in N[P] \setminus P$. So, we have that $v(k-1,l) \in F$ and $v(k,l-1) \in F$. By Corollary 1.9, $Z+(0,1) \in G_{-1} \setminus N[P]$. In order to have $\deg_{(h(F))}(v) = 2$ we need to check that $e(k,k+1;l)$ and $e(k;l,l+1)$ do not belong to $h(F)$. See Figure 4.8 (a).

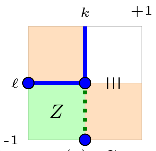

Fig. 4.8 (a). Case 2.1.

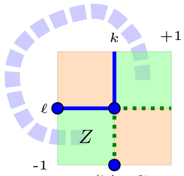

Fig. 4.8 (b). Case 2.1. $e(k,k+1;l) \in hb(F)$.

Assume for contradiction that $e(k,k+1;l)$ is a shadow edge of F. By Observation 4.2, $Z+(1,1) \in N[P] \setminus P$. Since F is an H-subtree, there is an H-path $P(Z, Z+(1,1))$ contained in F. But then $P(Z, Z+(1,1)), Z+(1,0), Z$ is an H-cycle, which contradicts Proposition 1.3. Thus $e(k,k+1;l)$ is not a shadow edge of F. It follows that $Z+(1,1)$ belongs to $G_{-1} \setminus F$. See Figure 4.8 (b).

Similarly, if $e(k;l,l+1) \in h(F)$, then by Corollary 1.9, $Z+(1,1) \in N[P]$ again, and we obtain the same contradiction as above. End of Case 2.1.

CASE 2.2: $e(k-1,k;l) \in H$ *and* $e(k,k+1;l) \in H$. Then $e(k;l,l+1) \notin H$. By Observation 4.2, exactly one of Z and $Z+(1,0)$ belongs to $N[P] \setminus P$ and the other belongs to $G_{-1} \setminus N[P]$. By symmetry, we may assume WLOG that $Z \in N[P] \setminus P$ and $Z+(1,0) \in G_{-1} \setminus N[P]$. By Corollary 1.9, $Z+(0,1) \in G_{-1} \setminus N[P]$. In order to have $\deg_{h(F)}(v) = 2$ we need to check that $e(k;l,l+1)$ and $e(k,k+1;l)$ do not belong to $h(F)$.

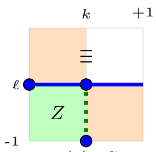

Fig. 4.9 (a). Case 2.2.

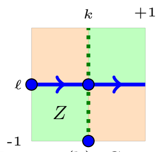

Fig. 4.9 (b). Case 2.2. $e(k;l,l+1) \in hb(F)$.

Assume for contradiction that $e(k;l,l+1)$ is a shadow edge of F. Then $Z+(1,1) \in F$. But now, if we orient H as directed cycle $\overrightarrow{K}_H$, Z and $Z+(1,1)$ are on different sides of $\overrightarrow{K}_H$, contradicting Corollary 1.11. Therefore, $e(k;l,l+1)$ cannot be a shadow edge of F. See Figure 4.9 (b). It follows that $Z+(1,1) \in G_{-1} \setminus N[P]$.

Similarly, if $e(k,k+1;l) \in h(F)$, then by Corollary 1.9, $Z+(1,1) \in N[P]$ again, and we obtain the same contradiction as above. End of Case 2.2. End of Case 2. End of proof for (i).

Proof of (ii). Let u, v be vertices in F. Orient the subpath $P = P(u,v)$ of H from u to v, labelled $u = u_0, u_1, \ldots$. If $E(P) \subset E(h(F))$, then we're done. Otherwise, let s be the number of shadow edges of $h(F)$ that are incident on P. Let u_{i_1} be the first vertex of P after u such that $u_{i_1} \in F$ but $u_{i_1+1} \notin F$. Let $u_{i'_1}$ be the first vertex of P after u_{i_1} that is in F. For $j \in \{2, \ldots, s\}$ let u_{i_j} be the first vertex of P after $u_{i'_{j-1}}$ such that $u_{i_j} \in F$ but $u_{i_j+1} \notin F$, and let $u_{i'_j}$ be the first vertex of P after u_{i_j} that is in F. Note that $P(u_{i'_s}, v) \subset P(u,v)$ and that $P(u_{i'_s}, v)$ is contained in F.
We claim that for $1 \leq j \leq s$, $\{u_{i_j}, u_{i'_j}\}$ is a shadow edge of F. It follows from this claim that $P(u, u_{i_1}), (u_{i_1}, u_{i'_1}), P(u_{i'_1}, u_{i_2}), (u_{i_2}, u_{i'_2}), \ldots, P(u_{i'_{s-1}}, u_{i_s}), (u_{i_s}, u_{i'_s}), P(u_{i'_s}, v)$ is contained in F. It remains to check that the claim is true.
Proof of Claim. For definiteness let $u_{i_j} = v(k,l)$, $u_{i_j+1} = v(k+1,l)$. Let $R(k-1,l-1) = Z$. Since u_{i_j+1} is not in F, $Z+(1,0)$ and $Z+(1,1)$ belong to $G_{-1} \setminus F$. Since u_{i_j} is in F, at least one of Z and $Z+(0,1)$ is a box of F.

We will first show that $u_{i_j-1} = v(k-1,l)$. Assume for contradiction that $u_{i_j-1} = v(k,l-1)$ or $u_{i_j-1} = v(k,l+1)$. By symmetry we only need to check one of the two. For definiteness assume that $u_{i_j-1} = v(k,l-1)$.

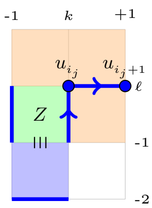

Fig. 4.10 (a).

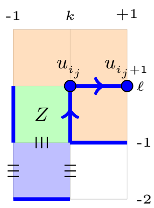

Fig. 4.10 (b).

Note that if $Z+(0,1) \in F$, by Observation 4.2, $Z+(0,1) \in N[P(X,Y)] \setminus P(X,Y)$. But then $Z+(0,1)$ is neither a leaf nor a switchable box, contradicting Lemma 4.1 (d). It remains to check the case where $Z+(0,1) \in G_{-1} \setminus F$ and $Z \in N[P(X,Y)] \setminus P(X,Y)$. See Figure 4.10 (a). Using Lemma 4.1 (d) again, we have that $Z+(0,-1) \in P(X,Y)$, $e(k-1;l-1,l) \in H$ and $e(k-1,k;l-1) \notin H$. By Lemma 4.1 (b) and (c), we have that $e(k-1,k;l-2) \in H$. Note that if $Z+(0,-1)$ is not an end-box of $P(X,Y)$, by Lemma 4.1 (b), $e(k,k+1;l-1) \in H$, $e(k-1;l-2,l-1) \notin H$ and $e(k;l-2,l-1) \notin H$. But then after $Z \mapsto Z+(1,0)$, $Z+(0,-1) \in P(X,Y)$ is switchable, contradicting that F is a looping fat path. See Figure 4.10(b) It remains to check the case where $Z+(0,-1)$ is an end-box of Z. WLOG assume that $Z+(0,-1) = X$. There are three possibilities: $X = Y+(2,0)$, $X = Y+(0,-2)$, and $X = Y+(-2,0)$.

CASE 1: $Y = X+(2,0)$. Then F must be northern, so $e(k+1;l-2,l-1) \in H$ and $e(k+1;l-1,l) \in H$. But then there is an H-cycle $P(Y,X), X+(0,1), X+(0,2), X+(1,2), X+(2,2), X+(2,1), Y$, which contradicts Proposition 1.3. See Figure 4.11 (a). End of Case 1.

CASE 2: $Y = X + (0,-2)$. Then F can be western or eastern.

CASE 2.1: F is western. Then $e(k-2,k-1;l-2) \in H$ and $e(k;l-2,l-1) \in H$. It follows that $X+(-1,0) \in P(X,Y)$ and that $e(k-2;l-2,l-1) \notin H$. But then $X+(-1,0)$ is switchable, contradicting that F is a looping fat path. See Figure 4.11 (b). End of Case 2.1.

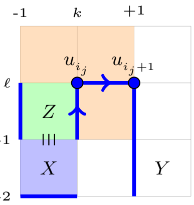

Fig. 4.11 (a). Case 1.

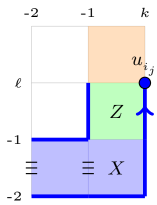

Fig. 4.11 (b). Case 2.1.

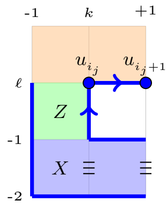

Fig. 4.11 (c). Case 2.2.

CASE 2.2: F is eastern. Then $e(k-1;l-2,l-1) \in H$ and $e(k,k+1;l-2) \in H$. It follows that $e(k,k+1;l-1) \in H$. But then $X+(1,0) \in P(X,Y)$ is switchable, contradicting that F is a looping fat path. See Figure 4.11 (c). End of Case 2.2. End of Case 2.

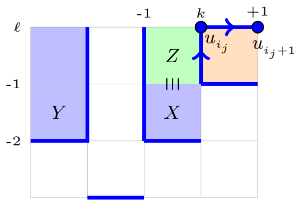

Fig. 4.12. Case 3.

CASE 3: $Y = X+(-2,0)$. Then F must be northern, so $e(k-1;l-2,l-1) \in H$. It follows that $e(k,k+1;l-1) \in H$. But then $X+(0,1) \mapsto X+(1,1)$, $X+(-1,0) \mapsto X$ is a short weakening, contradicting that F is a looping fat path. See Figure 4.12. End of Case 3. This concludes the proof that $u_{i_j-1} = v(k-1,l)$.

Now, by Corollary 1.9, exactly one of Z and $Z+(0,1)$ belongs to F. WLOG, assume that $Z \in F$. By Observation 4.2, since $Z+(1,0)$ is not in F, $Z \in N[P(X,Y)] \setminus P(X,Y)$. Note that this means that $e(k;l-1,l)$ is a shadow edge of F. By Lemma 4.1 (d), $e(k-1,k;l-1) \in H$, $e(k-1;l-1,l) \notin H$ and $Z+(-1,0) \in P(X,Y)$.

It remains to check that $u_{i'_j} = v(k,l-1)$. Assume for contradiction that $u_{i'_j} = v(a,b) \neq v(k,l-1)$. For definiteness assume that $u_{i'_j-1} = v(a+1,b)$. Note that, by the proof that $u_{i_j-1} = v(k-1,l)$, we have that $u_{i'_j+1} = v(a-1,b)$. Let $Z' = R(a-1,b)$. By Corollary 1.9, exactly one of Z' and $Z'+(0,-1)$ belongs to F. Note that if $Z' \notin F$, then $Z'+(0,-1) = \Phi((u_{i'_j}, u_{i'_j+1}), \text{left}) \in F$. But then, $Z \in F$ and $Z = \Phi((u_{i_j-1}, u_{i_j}), \text{right})$ contradicting Corollary 1.11. Thus we must have $Z' \in F$.

By Lemma 4.1 (d), we must have $e(a-1,a;b+1) \in H$, $e(a-1;b,b+1) \notin H$ and $Z'+(-1,0) \in P(X,Y)$. Let $P(Z+(1,0), Z'+(1,0))$ be the H-path from $Z+(1,0)$ to $Z'+(1,0)$ contained in the H-walk $\Phi(\overrightarrow{K}(((u_{i_j}, u_{i_j+1}), (u_{i'_j-1}, u_{i'_j})), \text{right}))$. Observe that $P(Z+(1,0), Z'+(1,0)) \subset G_{-1} \setminus F$. Since F is an H-subtree, F is H-path-connected so there is an H-path $P(Z', Z)$ contained in F. But then, then $P(Z', Z), P(Z+(1,0), Z'+(1,0))$ is an H-cycle, which contradicts Proposition 1.3. See Figure 4.13. Thus we must have $u_{i'_j} = v(k,l-1)$. □

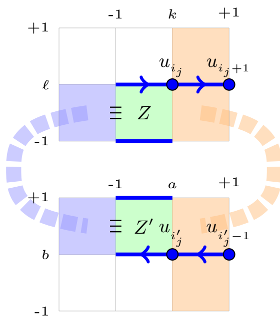

Fig. 4.13.

Proposition 4.4. Let G be an $m \times n$ grid graph, let H be a Hamiltonian cycle of G and let $F = G\langle N[P(X,Y)]\rangle$ be a looping fat path in G following a leaf L. Then $h(F)$ does not have consecutive colinear edges other than the left and right collinear edges in the A_1-type of F following L.

Proof. We need to check that $h(F)$ does not have consecutive colinear edges in the case where one of those colinear edges is a left or right colinear edge of the A_1-type of F and in the case where neither of those colinear edges is a left or right colinear edge of the A_1-type of F. We divide the proof into Lemmas 4.5 and 4.6.

Lemma 4.5. The shadow of F does not have a pair of consecutive colinear edges in the case where one edge of the pair is one of the left or right colinear edges of the A_1-type that follows L.

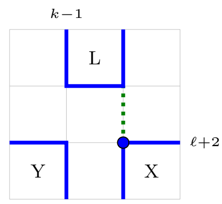

Fig. 4.14.

Proof. For definiteness, assume that F is a southern looping fat path following the leaf $R(k-1,l+3)$. See Figure 4.14. Assume for contradiction that $h(F)$ does have consecutive colinear edges and one of those edges is one of the right or left colinear edges of the A_1-type that follows $e(k-1,k;l+3)$. For definiteness, assume that one of those consecutive colinear edges is one of the right colinear edges of the A_1-type that follows L. Then, either $e(k;l+1,l+2)$, $e(k;l+2,l+3)$ is a pair of consecutive colinear edges or $e(k;l-1,l)$, $e(k;l,l+1)$ is a pair of consecutive colinear edges. If the former, then $\deg_{(h(F))}(v(k,l+2)) = 3$ contradicting Lemma 4.3, so we only need to check the latter. Suppose then, that $e(k;l-1,l)$, $e(k;l,l+1)$ is a pair of consecutive colinear edges. Note that X and $X+(0,-1)$ belong to F. If the edge $e(k;l-1,l)$ is in $h(F)$ then it is either a shadow edge or it belongs to H. We show that both cases lead to contradictions.

CASE 1: $e(k; l-1, l) \in H$. Then exactly one of $X + (0, -1)$ and $X + (1, 0)$ must belong to P.

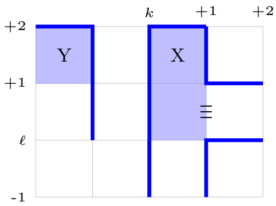

Fig. 4.15 (a). Case 1.1.

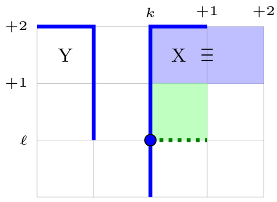

Fig. 4.15 (b). Case 1.2.

CASE 1.1: $X + (0, -1) \in P$. Note that if $e(k + 1; l, l+1) \in H$, then $X + (0, -1) \in P$ is switchable, contradicting the definition of a fat path, so we only need to check the case where $e(k + 1; l, l + 1) \notin H$. Then $S_{\uparrow}(k + 1, l - 1; k + 2, l) \in H$ and $S_{\downarrow}(k+1, l+2; k+2, l+1) \in H$. Now exactly one of $X + (0, -2)$ and $X + (1, -1)$ belong to P. Suppose that $X + (0, -2) \in P$ (Figure 4.15 (a)). Then $e(k, k+1; l-1) \in H$ or $e(k, k+1; l-1) \notin H$. If the former, then $X + (-2, 0)$ must be an end-box of P, but this contradicts the fact that the other end-box of P is $Y = X + (-2, 0)$; and if the latter then $X + (0, -2)$ is switchable, contradicting that F is a looping fat path. The case where $X + (1, -1)$ belongs to P is very similar so we omit the proof. End of Case 1.1

CASE 1.2: $X + (1, 0) \in P$. Then $X + (0, -1) \in N[P] \setminus P$ and $X + (0, -2) \notin N[P]$. It follows that $e(k, k+1; l)$ is a shadow edge of F. But then $\deg_{(h(F))}(v(k, l)) = 3$ contradicting Lemma 4.3. See Figure 4.15 (b). End of Case 1.2.

CASE 2: $e(k; l-1, l) \in hb(F)$. Let $Z = R(k-1, l-1)$. Then exactly one of Z and $Z + (1, 0)$ belongs to $N[P] \setminus P$.

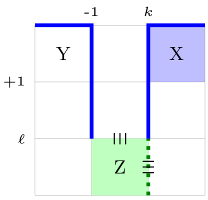

Fig. 4.16 (a). Case 2.
$Z \in N[P] \setminus P$.

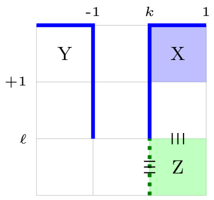

Fig. 4.16 (b). Case 2.
$Z + (1, 0) \in N[P] \setminus P$.

If $Z \in N[P] \setminus P$ then Lemma 4.3 implies that $e(k - 1, k; l) \notin E(h(F))$, but this contradicts Lemma 4.1 (d) (Figure 4.16 (a)); and if $Z + (1, 0) \in N[P] \setminus P$, then Lemma 4.3 implies that $e(k, k+1, l) \notin H$, but this also contradicts to Lemma 4.1 (d) (Figure 4.16 (b)). End of Case 2. □

Lemma 4.6. The shadow of F does not have a pair of consecutive colinear edges in the case where neither edge of the pair is a left or right colinear edge of the A_1-type that follows L.

Proof. It is a fact that none of the boxes of P discussed in this lemma can be end-boxes of P. The justifications are straightforward but distracting so we will omit them and use this fact repeatedly and implicitly throughout the proof.

Assume for contradiction that there is a pair of consecutive collinear edges in $h(F)$ where neither edge of the pair is a left or right colinear edge of the A_1-type that follows L. For definiteness, we may assume that these edges are the horizontal edges $e(k'-1, k'; l')$ and $e(k', k'+1; l')$. Let $R(k'-1, l') = Z$. Corollary 1.9 implies that exactly one of Z and $Z + (0, -1)$ is in F. For definiteness, assume that $Z \in F$. Then, by Lemma 4.3, $e(k'; l', l'+1)$ is neither in H nor in $hb(F)$. Now, either both $e(k'-1, k'; l')$ and $e(k', k'+1; l')$ belong to H, or at least one of them belongs to $hb(F)$.

CASE 1: Both $e(k'-1, k'; l')$ *and* $e(k', k'+1; l')$ *belong to* H. By Lemma 4.3, $e(k'; l', l'+1)$ is not a shadow edge of F. Then, Lemma 4.1 (a) implies that either exactly one of Z and $Z + (1, 0)$ belongs to $N[P] \setminus P$ or neither does.

CASE 1.1: Neither Z *nor* $Z + (1, 0)$ *belongs to* $N[P] \setminus P$. This implies that Z and $Z + (1, 0)$ are in P (Figure 4.17 (a)). Now, at least one of $e(k'-1, k'; l'+1)$ and $e(k', k'+1; l'+1)$ belongs to H. For definiteness assume that $e(k'-1, k'; l'+1) \in H$. But then Z is a switchable box in P, contradicting the definition of a fat path. End of Case 1.1

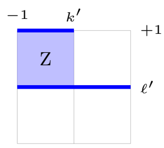

Fig. 4.17 (a).
Case 1.1.

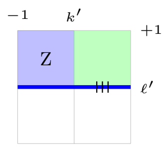

Fig. 4.17 (b).
Case 1.2.

CASE 1.2: Exactly one of Z *and* $Z + (1, 0)$ *belongs to* $N[P] \setminus P$. For definiteness, assume that $Z + (1, 0) \in N[P] \setminus P$. Then $Z \in P$. But now, the fact that Z is not an end-box and Lemma 4.1 (b) imply that $e(k', k'+1; l') \notin H$, contradicting the assumption of Case 1. End of Case 1.2. End of Case 1.

CASE 2: At least one of $e(k'-1, k'; l')$ *and* $e(k', k'+1; l')$ *belongs to* $hb(F)$. Either $e(k'-1, k'; l') \in hb(F)$, or $e(k'-1, k'; l') \notin hb(F)$.

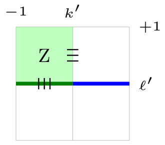

Fig. 4.18 (a) Case 2.1

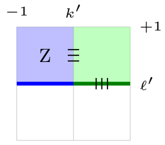

Fig. 4.18 (b). Case 2.2(a).

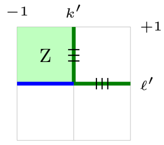

Fig. 4.18 (c). Case 2.2(b).

CASE 2.1: $e(k'-1,k';l') \in hb(F)$. It follows that $Z \in N[P] \backslash P$. But then Z can be neither switchable nor a leaf, contradicting Lemma 4.1(d). See Figure 4.18 (a). End of Case 2.1.

CASE 2.2: $e(k'-1,k';l') \notin hb(F)$. Then we must have that $e(k'-1,k';l') \in H$ and $e(k',k'+1;l') \in hb(F)$. Now, either $Z+(1,0) \in F$ or $Z+(1,0) \notin F$.

CASE 2.2(a): $Z+(1,0) \in F$. Then, by Observation 4.2, $Z+(1,0) \in N[P] \setminus P$. But $Z+(1,0)$ can be neither switchable nor a leaf, contradicting Lemma 4.1(d). See Figure 4.18 (b). End of Case 2.2(a).

CASE 2.2(b): $Z+(1,0) \notin F$. Then, Z must belong to $N[P] \setminus P$. This means that $e(k';l',l'+1) \in hb(F)$, which, as we, as we noted in the second paragraph of the proof, is not possible. See Figure 4.18 (c). End of Case 2.2(b). □

This completes the proof of Proposition 4.4. An immediate consequence of it is that the A_1-type following L is the only A_1-type in F, so we can refer to it as *the* A_1-type of F.

4.2 Turns

In this section we show that every looping fat path F must have a turn. We do this by showing that the shadow $h(F)$ of a looping fat path F must have a (necessarily closed) turn and note that this would immediately imply that F must have a turn.

In Lemma 4.7, Lemma 4.8, and Corollaries 4.9 (a) and (b) below, we will often use the definition of the shadow of southern looping fat path, Proposition 4.4, and the fact that the A_1-type of F is unique in F, and write (DsFP) whenever we appeal to them.

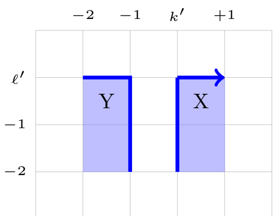

Fig. 4.19.

Lemma 4.7. Let H be a Hamiltonian cycle of an $m \times n$ grid graph G and let $h(F)$ be the shadow of a looping fat path of G. Then $h(F)$ has at least one turn T_1 such that both leaves of T_1 belong to F.

Proof. For definiteness assume that $F = G\langle N[P(X,Y)]\rangle$ is a southern looping fat path following $R(k'-1,l'+1)$ southward, with $X = R(k',l'-1)$ and $Y = R(k'-2,l'-1)$. By Lemma 4.3, $E(h(F))$ is a cycle. Orient $E(h(F))$ as a directed trail $\overrightarrow{K}$ so that the first edge of $\overrightarrow{K}$ is $(v(k',l'),v(k'+1,l'))$. With this orientation we can give a direction - N,S,E or W - to edges in $\overrightarrow{K}$, defined as the position of the head of an edge relative to its tail. See Figure 4.19.

Our choice of direction for the first edge and Lemma 1.6 imply that $\text{Boxes}(\Phi(\overrightarrow{K},\text{right})) \subset \text{Boxes}(F)$ so the boxes of $h(F)$ are on the right side of the oriented edges of K. We call this fact (RSK) for reference. We sweep the edges of K in the direction of the orientation starting at $v(k',l')$. We observe that we must encounter at least one west edge e_W, since Y is west of X. Let $e_W = e_0 = e(k-1,k;l)$ be the first west edge encountered, let e_1 be the edge preceding e_0 in the sweep, let e_j be the edge preceding the edge e_{j-1} in the sweep and let $(v(k',l'),v(k'+1,l')) = e_s$.

In this proof we will use the fact that e_W is the first west edge encountered (1stW) several times.

By (DsFP), e_0 was immediately preceded by a south edge or a north edge.

CASE 1: e_1 is southern. We shall find a northeastern turn. By (DsFP) and (1stW), the preceding edge e_2 must be eastern; By (DsFP) and (1stW), e_3 has to be southern. By (1stW) e_4 cannot be western. Then e_4 is southern or e_4 is eastern.

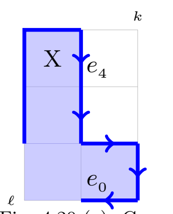

Fig. 4.20 (a). Case 1.1. $\Phi(e_4,\text{right}) = X$.

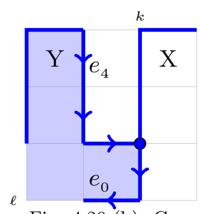

Fig. 4.20 (b). Case 1.1. $\Phi(e_4,\text{right}) = Y$.

CASE 1.1: e_4 is southern. Then, by (DsFP) and (RSK), we have that $\Phi(e_4,\text{right}) = X$ or $\Phi(e_4,\text{right}) = Y$. But the former contradicts Proposition 4.4, and the latter implies that $\deg_{h(F)}(v(k,l+1) = 3$, contradicting Lemma 4.3. See Figure 4.20. Thus, e_4 must be eastern. End of Case 1.1.

CASE 1.2: e_4 is eastern. By (DsFP), e_5 is not eastern. Then e_5 is northern or e_5 is southern.

CASE 1.2(a): e_5 is northern. Then there is a northeastern turn T_1 on the edges $e_0, ..., e_5$ with both leaves contained in F. See Figure 4.21 (a). End of Case 1.2 (a).

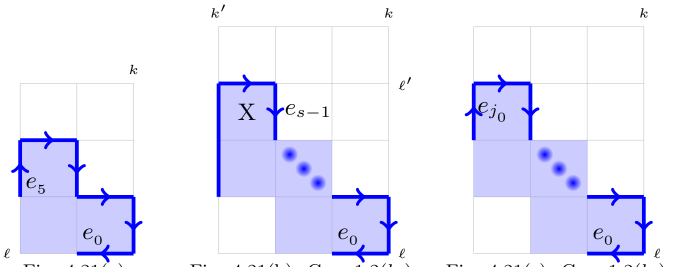

Fig. 4.21(a). Fig. 4.21(b). Case 1.2(b_1). Fig. 4.21(c). Case 1.2(b_2).

CASE 1.2 (b): e_5 is southern. Let $Q(j)$ be the statement: "e_j is southern and e_{j+1} is eastern". Now, either $Q(j)$ is true for each $j \in \{1, 3, ..., s-1\}$ (Case 1.2(b_1)), or there is some $j_0 \in \{5, 9, ..., s-1\}$ such that $Q(j)$ for each odd $j < j_0$, but $Q(j_0)$ is not true (Case 1.2(b_2)).

CASE 1.2(b_1). Then we have a northeastern turn T_1 on the edges $e_0, ..., e_s, e(k'; l'-1, l')$ with both leaves contained in F. See Figure 4.21 (b). End of Case 1.2(b_1).

CASE 1.2(b_2). By (DsFP), e_{j_0} is not eastern. If e_{j_0} is southern, then we run into the same contradiction as in Case 1.1; and if e_{j_0} is northern then we have a northeastern turn T_1 on the edges $e_0, ..., e_{j_0}$ with both leaves contained in F (Figure 4.21 (c)). End of Case 1.2(b_2). End of Case 1.2 (b). End of Case 1.2.

CASE 2: e_1 is northern. We shall find a southeastern turn with both leaves contained in F. By (1stW) and (DsFP), e_2 is eastern. By (DsFP), e_3 is northern. Note that (RSK) and Lemma 4.1 (d) imply that $\Phi(e_1, \text{right}) \in P$ and $e_1 \in H$. By Lemma 4.3, $e(k, k+1; l-1) \notin H$ and $e(k, k+1; l) \notin H$. If $e(k+2; l-1, l) \in H$, then $\Phi(e_1, \text{right})$ is switchable and in P, contradicting the definition of a fat path, so we may assume that $e(k+2; l-1, l) \notin H$. Then we must have that $S_{\downarrow}(k+1, l+1; k+2, l) \in H$, $S_{\uparrow}(k+1, l-2; k+2, l-1) \in H$ and that $e(k+2; l-1, l) \in hb(F)$.

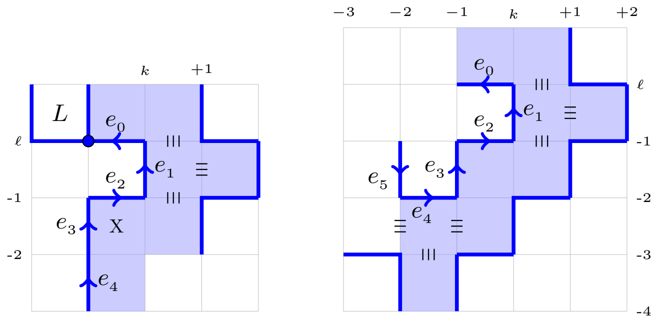

Fig. 4.22(a). Case 2. Fig. 4.22(b). Case 2.1

By (1stW), e_4 is not western. If e_4 is northern then (DsFP) and (RSK) imply that $\Phi(e_3, \text{right}) = X$. But then $L = R(k-1, l)$ and then $\deg_H(v(k-1, l)) = 3$, contradicting H is Hamiltonian. See Figure 4.22 (a). Then e_4 must be eastern. By (DsFP), e_5 is not eastern. Then e_5 is southern or northern.

CASE 2.1: e_5 is southern. By (RSK) and Lemma 4.1 (d), we have that $\Phi(e_4, \text{right}) \in P$, and that $e_4 \in H$. By Lemma 4.3, $e(k-2; l-3, l-2) \notin H$ and $e(k-1; l-3, l-2) \notin H$. If $e(k-2, k-1; l-3) \in H$, then $\Phi(e_4, \text{right})$ is switchable and in P, contradicting that F is a fat path, so we may assume that $e(k-2, k-1; l-3) \notin H$. It follows that $S_{\rightarrow}(k-3, l-3; k-2, l-4) \in H$ and that $S_{\uparrow}(k-1, l-4; k+2, l-1) \in H$. Then there is a southeastern turn on $e(k-2; l-4, l-3)$, $S_{\uparrow}(k-1, l-4; k+2, l-1) \in H$, $e(k+1, k+2; l)$ with both leaves in F. See Figure 4.22 (b). End of Case 2.1

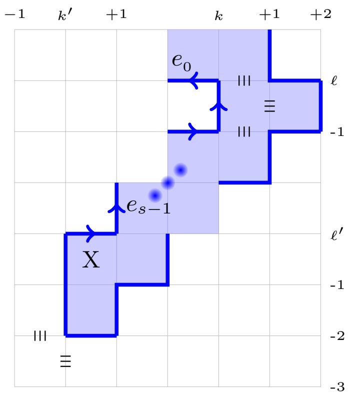

Fig. 4.23. Case 2.2(a).

CASE 2.2: e_5 is northern. Let $Q(j)$ be the statement: "e_j is northern and e_{j+1} is eastern". Now, either $Q(j)$ is true for each $j \in \{1, 3, ..., s-1\}$ (Case 2.2(a)), or there is some $j_0 \in \{5, 7, ..., s-1\}$ such that $Q(j)$ for each odd $j < j_0$, but $Q(j_0)$ is not true (Case 2.2(b)).

CASE 2.2(a). Then (DsFP) and (RSK) imply that $\Phi(e_s, \text{right}) = X$. This means that $e(k'; l'-2, l'-1) \in H$, $e(k'; l'-1, l') \in H$, and that $\Phi(e_{s-1}, \text{right}) \in h(F)$. By Proposition 4.4, $e(k'; l'-3, l'-2) \notin H$. Note that if $e(k'-1, k'; l'-2) \in H$, then $P(X, Y)$ is the H-path $X, X+(0,-1), X+(0,-2), X+(-1,-2), X+(-2,-2), X+(-2,-1), Y$. This contradicts our finding that $\Phi(e_{s-1}, \text{right}) \in h(F)$. Then it must be the case that $e(k'-1, k'; l'-2) \notin H$. Then we must have $e(k', k'+1; l'-2) \in H$. It follows that $S_{\uparrow}(k'+1, l'-2; k+, l-1) \in H$. Then there is a southeastern turn T_1 on $e(k'; l'-2, l'-1)$, $S_{\rightarrow}(k'+1, l'-2; k+2, l-1)$, $e(k+1, k+2; l)$ with both leaves contained in F. See Figure 4.23. End of Case 2.2(a).

CASE 2.2(b). By (DsFP), e_{j_0} is not eastern. Suppose that e_{j_0} is northern (in orange in Figure 4.25). The assumption that $Q(j_0)$ is false implies that e_{j_0+1} is not eastern and (1stW) implies that e_{j_0+1} is not western. It must be the case that e_{j_0+1} is also northern. By (DsFP) and (RSK), we have that $\Phi(e_{j_0}, \text{right}) = X$. But then $j_0 - 1 = s$, contradicting the assumption that $j_0 \in \{5, ..., s-1\}$ (in orange in Figure 4.24). Thus e_{j_0} cannot be northern. It follows that e_{j_0} is southern. Let $e_{j_0} = e(k''; l'', l''+1)$.

Using the same arguments as in Case 2.1, we find that $e(k''; l''-1, l'') \notin H$, $e(k''+1; l''-1, l'') \notin H$, $e(k'', k''+1; l''-1) \notin H$, and that $\Phi(j_0 - 1, \text{right}) \in P$. It follows that $S_{\to}(k''-1, l''-1; k'', l''-2) \in H$ and that $S_{\uparrow}(k''+1, l''-2; k+2, l-1) \in H$. Then there is a southeastern turn T_1 on $e(k''; l''-2, l''-1)$, $S_{\uparrow}(k''+1, l''-2; k+2, l-1)$, $e(k+1, k+2; l)$. with both leaves contained in F (Figure 4.25). End of Case 2.2(b). End of Case 2.2. End of Case 2. □

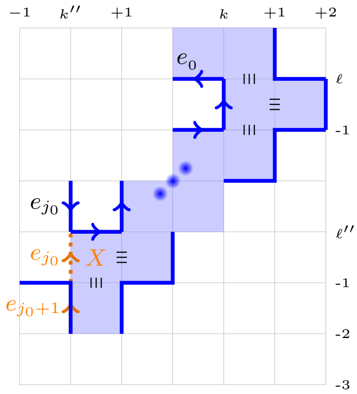

Fig. 4.24. Case 2.2(b).

Definition. Let G be an $m \times n$ grid graph, let H be a Hamiltonian cycle of G, and let F be a looping fat path in G. We say that a turn T of $h(F)$ is *admissible* if:

(i) no leaf of T is an end-box of F, and

(ii) both leaves of T belong to F.

Lemma 4.8. Let H be a Hamiltonian cycle of an $m \times n$ grid graph G and let $h(F)$ be the shadow of a looping fat path of G. Then $h(F)$ has an admissible turn.

Proof. Let F, X, Y, $\overrightarrow{K}$ and $e_W, e_1, ...e_s$ be as in Lemma 4.7, including the assumption that F is southern and (RSK).

CASE 1: e_1 is southern. By Case 1 in Lemma 4.7, there is a northeastern turn T_1. We continue sweeping $\overrightarrow{K}$, beginning from e_W, until we find the first northern edge e_N in the subtrail $\overrightarrow{K}(e_W, e_N)$ of $\overrightarrow{K}$, where $e_N = (\widehat{k}; \widehat{l}, \widehat{l}+1) = \widehat{e_0}$. We write (1stN) to refer to the fact that e_N is the first northern edge encountered after e_W, whenever we appeal to it. Let $\widehat{e_1}$ be the edge preceding $\widehat{e_0}$ in the sweep, let $\widehat{e_j}$ be the edge preceding the edge $\widehat{e_{j-1}}$ in the sweep and let $\widehat{e_{t+1}} = e_W$. Then $\widehat{e_1}$ is western or $\widehat{e_1}$ is eastern.

Before we consider each case, we will check that the subtrail $\overrightarrow{K}(\widehat{e_0}, \widehat{e_t})$ of $\overrightarrow{K}$ does not contain the right or left colinear edges of the A_1-type of F. To this end, we will translate H by $(-k', -l')$ to simplify calculations. (DsFP) and (1stW) imply that for every eastern edge in the subtrail $\overrightarrow{K}(e_s, e_1)$ of $\overrightarrow{K}$ there is at most one northern or southern edge. Denote by v_{end} the head of the edge e_W. The assumption that e_1 is southern and the fact that a shortest turn has length two imply that v_{end} is contained in the region $U_{1,\text{end}}$, determined by $x \geq 1$ and $|y+2| \leq x-1$ (Eq.1). Let $v_{\text{end}} = v(a, b)$. It follows that $\overrightarrow{K}(\widehat{e_t}, \widehat{e_0})$ is contained in the region U_2 bounded by $y \leq b+1$ and $|x-a| \leq b+1-y$ (Eq.2). See Figure 4.35.

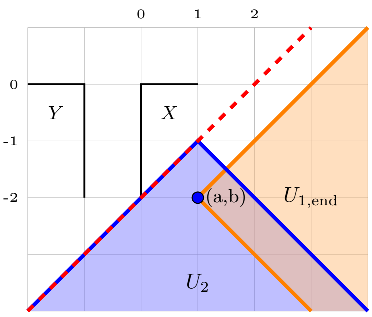

Fig. 4.25. Case 3.2(a) The line $y - x + 2 = 0$ in red; $U_{1,\text{end}}$ shaded orange, U_2 shaded blue.

We will check that U_2 and the colinear edges of the A_1-type of F lie on two different sides of the line $y = x - 2$. By (Eq.1) we have that $b \leq a - 3$ and by (Eq.2) we have that $y \leq x - a + b + 1$. Let $(x, y) \in U_2$. Then $y - x + 2 \leq -a + b + 3 \leq 0$, so U_2 lies below the line $y - x + 2$. Plugging in the values of the coordinates of the vertices A_1-type $v(x_1, y_1)$, we see that they lie above the line $y = x - 2$. This shows that $\overrightarrow{K}(\widehat{e_t}, \widehat{e_0})$ does not contain colinear edges. We will write (NCE) whenever we appeal to this fact. Note that (NCE) implies (i).

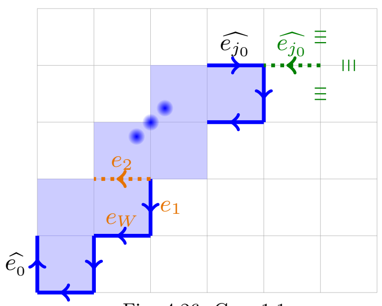

Fig. 4.26. Case 1.1.

CASE 1.1: $\widehat{e_1}$ is western. Note that $\widehat{e_1} \neq e_W$, otherwise we get a cycle on e_1, e_W, e_1, e_2. By (DsFP), $\widehat{e_2}$ is not western and by (1stN), $\widehat{e_2}$ is not northern, so $\widehat{e_2}$ must be southern. By (NCE) $\widehat{e_3}$ cannot be southern. Then $\widehat{e_3}$ must be western. If $\widehat{e_3} = e_W$, then we have a southeastern turn T_2 on $\widehat{e_0}, ..., \widehat{e_3}, e_1, e_2$, satisfying (i) and, by (RSK), (ii) (in orange in Figure 4.27), so we may assume that $\widehat{e_3} \neq e_W$. By (DsFP), $\widehat{e_4}$ is not western and by (1stN), $\widehat{e_4}$ is not northern. Then $\widehat{e_4}$ is southern.

Let $Q(j)$ be the statement: "$\widehat{e_j}$ is western and $\widehat{e_{j+1}}$ is southern".

Now, either $Q(j)$ is true for each $j \in \{1,3,...,t+1\}$, or there is some $j_0 \in \{5,7,...,t+1\}$ such that $Q(j)$ for each odd $j < j_0$, but $Q(j_0)$ is not true. If the former then we have a southeastern turn T_2 on $\widehat{e_0}, \widehat{e_1}, ..., \widehat{e_t}, e_W, e_1, e_2$ satisfying (i) and (ii) (blue in Figure 4.26), so assume the latter. By (NCE), $\widehat{e_{j_0}}$ is not southern. If $\widehat{e_{j_0}}$ is eastern then we have southeastern turn on $\widehat{e_0}, \widehat{e_1}, ..., \widehat{e_{j_0}}$ satisfying (i) and (ii) (blue in figure 4.26). Suppose then, that $\widehat{e_{j_0}}$ is western (green in figure 4.26). This is impossible: by (1stN). $\widehat{e_{j_0+1}}$ is not northern; since $Q(j_0)$ is false, $\widehat{e_{j_0+1}}$ is not southern; and by (DsFP), $\widehat{e_{j_0+1}}$ is not western. End of Case 1.1

CASE 1.2: $\widehat{e_1}$ is eastern. By (DsFP), $\widehat{e_2}$ is not eastern and by (1stN), $\widehat{e_2}$ is not northern, so $\widehat{e_2}$ must be southern. By (NCE) $\widehat{e_3}$ is not southern and by (DsFP), $\widehat{e_3}$ is not western. Then $\widehat{e_3}$ must be eastern. (DsFP) and (1stN) imply that $\widehat{e_4}$ must be southern.

Let $Q(j)$ be the statement: "$\widehat{e_j}$ is eastern and $\widehat{e_{j+1}}$ is southern". Now, either $Q(j)$ is true for each $j \in \{1,3,...,t-1\}$, or there is some $j_0 \in \{5,7,...,t-1\}$ such that $Q(j)$ is true for each odd $j < j_0$, but $Q(j_0)$ is not true.

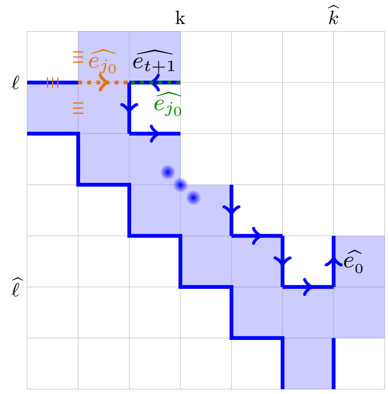

Fig. 4.27. Case 1.2 (a) and (b)

CASE 1.2 (a): $Q(j)$ is true for each $j \in \{1,3,...,t-1\}$. Then we have a southwestern turn on $\widehat{e_0}, \widehat{e_1}, ..., \widehat{e_{t-1}}, \widehat{e_t}, e_W$. Recall that $e_W = e(k-1,k;l)$. Observe that $R(k-2;l-1) \in P$, so $e(k-2;l-1,l) \notin H$. Similarly, $R(\widehat{k}-1,\widehat{l}-1) \in P$ and $e(\widehat{k}-1,\widehat{k};\widehat{l}-1) \notin H$. It follows that $R(k-3,l-1) \in F$, $R(\widehat{k}-1,\widehat{l}) \in F$ and that there is a southeastern turn T_2 on $e(k-3,k-2;l)$, $S_{\rightarrow}(k-3,l-1;\widehat{k}-1,\widehat{l}-2)$, $e(\widehat{k};\widehat{l}-2;\widehat{l}-1)$ satisfying (i) and (ii) (blue in Figure 4.27). End of Case 1.2 (a)

CASE 1.2 (b): There is some $j_0 \in \{5,7,...,t-1\}$ such that $Q(j)$ is true for each odd $j < j_0$, but $Q(j_0)$ is not true. If $\widehat{e_{j_0}}$ is western then we have a southeastern turn on $\widehat{e_0}, \widehat{e_1}, ..., \widehat{e_{j_0}}$. Then as in Case 1.2 (a), there is a southeastern turn T_2 satisfying (i) and (ii) (in blue in Figure 4.27, with e_{j_0} in green).

By (DsFP), $\widehat{e_{j_0}}$ is not southern. Suppose then $\widehat{e_{j_0}}$ is eastern. This is impossible: by (1stN). $\widehat{e_{j_0+1}}$ is not northern; since $Q(j_0)$ is false, $\widehat{e_{j_0+1}}$ is not southern; and by (DsFP), $\widehat{e_{j_0+1}}$ is not eastern (in orange in Figure 4.27). End of Case 1.2 (b). End of Case 1.2.

CASE 2: e_1 is northern. By Case 2 in Lemma 4.7, $\overrightarrow{K}$, has a southeastern turn T_1. We continue sweeping K, beginning from e_W, until we find the first southern edge e_S in the subtrail $\overrightarrow{K}(e_W, e_S)$ of $\overrightarrow{K}$. where $e_S = (\widehat{k};\widehat{l},\widehat{l}+1) = \widehat{e_0}$. We write (1stS) to refer to the fact that e_S is the first southern edge encountered after e_W, whenever we appeal to it. Let $\widehat{e_1}$ be the edge preceding $\widehat{e_0}$ in the sweep, let $\widehat{e_j}$ be the edge preceding the edge $\widehat{e_{j-1}}$ in the sweep and let $\widehat{e_{t+1}} = e_W$. Then $\widehat{e_1}$ is western or $\widehat{e_1}$ is eastern.

CASE 2.1: $\widehat{e_1}$ is western. Note that the assumption that e_1 is northern implies that $\widehat{e_1} \neq e_W$, otherwise there is a cycle $e_2, e_1, e_W, \widehat{e_0}$. By (DsFP), $\widehat{e_2}$ is not western and by 1stS, $\widehat{e_2}$ is not southern, so $\widehat{e_2}$ must be northern. By (DsFP)$\widehat{e_3}$ is not northern or eastern. Then $\widehat{e_3}$ must be western. By 1stS $\widehat{e_4}$ is not southern, and by (DsFP), $\widehat{e_4}$ is not western. Then $\widehat{e_4}$ must be northern. Let $Q(j)$ be the statement: "$\widehat{e_j}$ is western and $\widehat{e_{j+1}}$ is northern. Then either $Q(j)$ is true for each $j \in \{1,3,...,t-1\}$ or there is some $j_0 \in \{5,7,...,t-1\}$ such that $Q(j)$ for each odd $j < j_0$, but $Q(j_0)$ is not true.

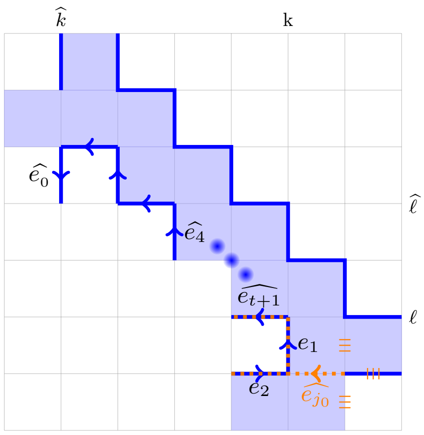

Fig. 4.28. Case 2.1 (a) and (b).

CASE 2.1 (a): $Q(j)$ is true for each $j \in \{1,3,...,t-1\}$. Then there is a northeastern turn on $\widehat{e_0}, \widehat{e_1}, ..., \widehat{e_{t-1}}, \widehat{e_t}, e_W, e_1, e_2$. As in Case 1.2 (a), we have that $R(\widehat{k},\widehat{l}+1) \in P$, $R(k,l-1) \in P$, $e(\widehat{k},\widehat{k}+1;\widehat{l}+2) \notin H$ and $e(k+1;l-1,l) \notin H$. Then there is a northeastern turn T_2 on $e(\widehat{k};\widehat{l}+2,\widehat{l}+3)$, $S_{\downarrow}(\widehat{k}+1,\widehat{l}+3;k+2,l)$, $e(k+1,k+2;l-1)$ satisfying (i) and (ii) (in blue in figure 4.28). End of Case 2.1(a).

CASE 2.1 (b): There is some $j_0 \in \{5,7,...,t-1\}$ such that $Q(j)$ for each odd $j < j_0$, but $Q(j_0)$ is not true. If $\widehat{e_{j_0}}$ is eastern then we have a northeastern turn on $\widehat{e_0}, \widehat{e_1}, ..., \widehat{e_{j_0}}$. Then, as in Case 2.1(a), there is a northeastern turn T_2 satisfying (i) and (ii).

By (DsFP), $\widehat{e_{j_0}}$ is not southern. Suppose then, that $\widehat{e_{j_0}}$ is western. This is impossible: by 1stS. $\widehat{e_{j_0+1}}$ is not southern; since

$Q(j_0)$ is false, $\widehat{e_{j_0+1}}$ is not northern; and by (DsFP), $\widehat{e_{j_0+1}}$ is not western (in orange in Figure 4.28). End of Case 2.1. End of Case 2.1(b). End of Case 2.1.

CASE 2.2: $\widehat{e_1}$ is eastern. By 1stS and (DsFP), $\widehat{e_2}$ is northern. By (DsFP), $\widehat{e_3}$ is not western. An argument analogous to (NCE-1) in Case 1 can be used to show that T_2 and the A_1-type lie on two different sides of the line $y = 2 - x$. In this case, we have that the region $U_{1,\text{end}}$ containing v_{end} is determined by $x \geq 1$ and $|y - 2| \leq x - 1$, and the region U_2 containing T_2, as defined in the next paragraph, is determined by $y \geq b - 1$ and $|x - a| \leq y - b + 1$. We will refer to this argument as (NCE-2). Note that by (NCE-2), $\widehat{e_3}$ is not northern. Then $\widehat{e_3}$ must be eastern. By (DsFP) and (1stS) $\widehat{e_4}$ is not southern or eastern. Then $\widehat{e_4}$ must be northern.

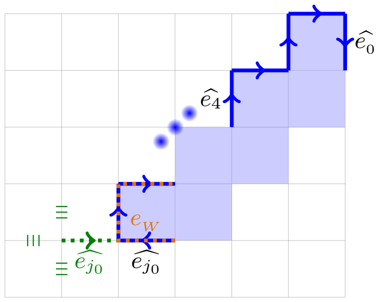

Fig. 4.29. Case 2.2.

Let $Q(j)$ be the statement: "$\widehat{e_j}$ is eastern and $\widehat{e_{j+1}}$ is northern". Now, either $Q(j)$ is true for each $j \in \{1, 3, ..., t-1\}$, or there is some $j_0 \in \{5, 7, ..., t-1\}$ such that $Q(j)$ for each odd $j < j_0$, but $Q(j_0)$ is not true. If the former then we have a northwestern turn T_2 on $\widehat{e_0}, \widehat{e_1}, ..., \widehat{e_{t-1}}, \widehat{e_t}, e_W$ satisfying (i) and (ii), (blue in figure 4.29 with $\widehat{e_{t-1}}, \widehat{e_t}, e_W$ dotted orange) so assume the latter. If $\widehat{e_{j_0}}$ is western then again we have a northwestern turn T_2 on $\widehat{e_0}, \widehat{e_1}, ..., \widehat{e_{j_0}}$ satisfying (i) and (ii) (blue in figure 4.29). By (NCE-2), $\widehat{e_{j_0}}$ is not southern. Then, suppose that $\widehat{e_{j_0}}$ is western (green in figure 4.33). This is impossible: by 1stS. $\widehat{e_{j_0+1}}$ is not southern; since $Q(j_0)$ is false, $\widehat{e_{j_0+1}}$ is not northern; and by (DsFP), $\widehat{e_{j_0+1}}$ is not eastern. End of Case 2.2.

Corollary 4.9 Let G be an $m \times n$ grid graph, let H be a Hamiltonian cycle of G, let $F = G\langle N[P(X,Y)]\rangle$ be a looping fat path in G, let T be an admissible turn of F, let L be a leaf of T, and let L' be the H-neighbour of L in F. Then:

(a) $d(T) \geq 3$, and

(b) $L \in N[P] \setminus P$ and $L' \in P$.

Proof of (a). We prove the contrapositive. For definiteness, assume that T is northeastern with northern leaf $L_N = R(k, l-1)$. Suppose that $d(T) < 3$. Then $d(T) = 2$ and the eastern leaf of T must be $L_E = R(k+1, l-2)$. Note that $L_N + (0,-1) \in F$, otherwise $L_E, ..., L_N, L_N + (0,-1)$ is an H-cycle.

Now, by Proposition 4.4, $e(k; l-2, l-1)$ and $e(k, k+1; l-2)$ cannot both belong to H. Then, either exactly one of $e(k; l-2, l-1)$ and $e(k, k+1; l-2)$ belongs to H, or neither does.

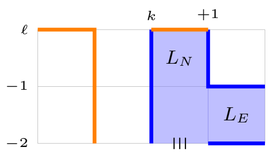

Fig. 4.30. Case 1.

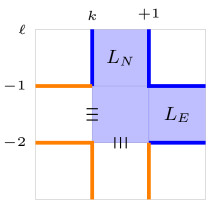

Fig. 4.31. Case 2.

CASE 1: exactly one of $e(k; l-2, l-1)$ and $e(k, k+1; l-2)$ belongs to H. By symmetry, we may assume WLOG that $e(k; l-2, l-1) \in H$ and $e(k, k+1; l-2) \notin H$. Note that the assumption that $e(k, k+1; l-2) \notin H$ implies that F is northern. It follows that L_N is an end-box of $P(X,Y)$. See figure 4.30. End of Case 1.

CASE 2: neither $e(k; l-2, l-1)$ nor $e(k, k+1; l-2)$ belongs to H. Then $e(k-1, k; l-1)$, $e(k+1; l-3, l-2)$ and $S_{\rightarrow}(k-1, l-2; k, l-3)$ belong to H. By Lemma 4.1(d), $L_N + (0,-1)$ must be long to $P(X,Y)$. It follows that at least one of L_N, $L_N + (0,-2)$, L_E and $L_E + (-2, 0)$ belongs to $P(X,Y)$ and is switchable, contradicting the assumption that F is a looping fat path. See figure 4.31. End of Case 2. End of proof for (a).

Proof of (b). Let T be an admissible turn. For definiteness, assume that T is northeastern with northern leaf $L_N = R(a,b)$ By Corollary 4.9(a), $d(T) \geq 3$. Then we have that $e(a, a+1; b-1) \in H$, and that $e(a+1; b-1, b) \notin H$. By (RSK), $L_N + (0,-1)$ and $L_N + (1,-1)$ belong to F. This means that $L_N + (0,-1) = L'_N$ is the H-neighbour of L_N in F. Now, either $e(a, a+1; b+1) \in H$ or $e(a, a+1; b+1) \notin H$. If $e(a, a+1; b+1) \in H$, then, since L_N is not an end-box of P, $L_N \in N[P] \setminus P$. Then, by Lemma 4.1 (b), $L'_N \in P$. And if $e(a, a+1; b+1) \notin H$, then L_N is switchable, so $L_N \in N[P] \setminus P$. Since $L'_N \in F$, by Lemma 4.1 (b), $L'_N \in P$. Either way we have that $L'_N \in P$ and and $L_N \in N[P] \setminus P$. See Figure 4.32. End of proof for (b). □

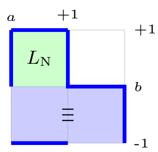

Fig. 4.32(a). $e(a, a+1; b+1) \in H$.

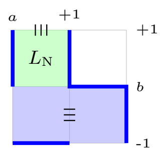

Fig. 4.32(b). $e(a, a+1; b+1) \notin H$.

4.3 Turn weakenings.

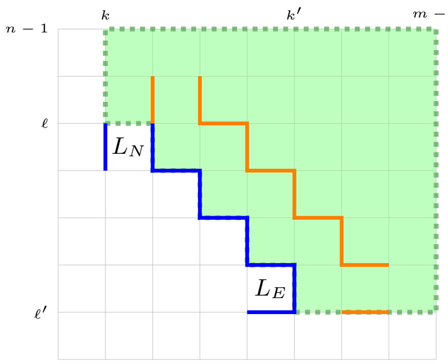

Fig. 4.33. A half-open turn T in blue, its lengthening T' in orange, Sector(T) shaded in green.

Definitions. Let H be a Hamiltonian cycle of an $m \times n$ grid graph G. Let T be a northeastern turn of H on $\{e(k; l-1, l), S_{\downarrow}(k+1, l; k', l'+1), e(k'-1, k'; l')\}$ and let L_N and L_E be the leaves of T. Define a *weakening* of T to be a cascade $\mu_1, ..., \mu_s$, where μ_s is the first valid nontrivial move in the cascade that has the form $L \mapsto L'$, or $L' \mapsto L$, with $L = L_N$ or $L = L_E$. We note that $\mu_1, ..., \mu_{s-1}$ must avoid T, as T has no switchable boxes or leaves, other than L_1 and L_2. We call a weakening consisting of three or less moves a *short weakening*. We call the subgraph $S_{\downarrow}(k+1, l; k', l'+1)$ *the stairs-part of* T and denote it by $S(T)$. We say that T has a *lengthening* T' if T' is a northeastern turn of H such that:

a) $d(T') \geq d(T)$ and

b) $S(T') \supseteq S(T) + (1, 1)$.

Analogous definitions apply to southeastern, southwestern and northwestern turns. We note that if T' is a lengthening of T, then T' is unique. Given a turn T_0 let $\mathcal{T}(T_0) = \mathcal{T}$ be a set of lengthenings such that:

1. The turn $T_0 \in \mathcal{T}$
2. The turn $T_j \in \mathcal{T}$ if and only if T_j is a lengthening of the turn T_{j-1}.

Define the *sector of* T to be the induced subgraph of G bounded by $e(k, k+1; l)$, $S_{\downarrow}(k+1, l; k', l')$, $e(k'-1, k'; l')$, and the segments $[(k', l'), (m-1, l')]$, $[(m-1, l'), (m-1, n-1)]$, $[(m-1, n-1), (k, n-1)]$, $[(k, n-1), (k, l)]$, and denote it by Sector(T). See Figure 4.33. Analogous definitions apply to sectors of southeastern, southwestern and northwestern turns.

Lemma 4.10. Let H be a Hamiltonian cycle of an $m \times n$ grid graph G, and let T be a turn in H with $d(T) \geq 3$. Then:

I. T has a short weakening or T has a lengthening.

II. If T' is a lengthening of T and T' has a weakening of length at most s, then T has a weakening of length at most $s+1$, with $s+1 \leq \min(m, n)$.

We prove Lemma 4.10 after we use it to prove Proposition 4.11.

Proposition 4.11. Let H be a Hamiltonian cycle of an $m \times n$ grid graph G, and let T be a turn in H with $d(T) \geq 3$. Then T has a weakening of length at most $\min(m, n)$.

Proof. Let $T = T_0$ be an admissible turn of H. If T_0 has a short weakening, then we're done, so we assume T_0 has no short weakening. By I in Lemma 4.10, T_0 has a lengthening T_1. So, $T_1 \in \mathcal{T}$, where $\mathcal{T} = \mathcal{T}(T_0)$. Since $m, n < \infty$, we have that $|\mathcal{T}| < \infty$. Let $\mathcal{T} = \{T_0, T_1, ..., T_j\}$. Then T_j has no lengthening; thus, by I of Lemma 4.10, it must have a short weakening. Then, by induction and II on Lemma 4.10, T_0 has a weakening. The bound follows immediately. □

Proof of Lemma 4.10. We first remark that none of the moves we use throughout this proof fit the description of the moves in Observation 3.4 (i) and (ii) in Section 3. We will use this fact repeatedly and implicitly.

Let H be a Hamiltonian cycle of G and let T be a turn of H with $d(T) \geq 3$. For definiteness, assume that T is northeastern and that T is on $\{e(k; l-1, l), S_{\downarrow}(k+1, l; k', l'+1), e(k'-1, k'; l')\}$. Let L_{N} be the northern leaf of T and let L_{E} be the eastern leaf of T. Since $d(T) \geq 3$, $m-1 \geq k+3$ and $0 \leq l-3$. L_{N} can be open or closed, so there are two cases to check.

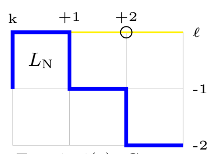

Fig. 4.34(a). Case 1

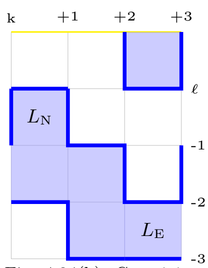

Fig. 4.34(b). Case 1.1.

Proof of CASE 1: L_N is closed. Proof of I. First we note $n-1 \neq l$, otherwise H misses $v(k+2, l)$. Then we must have $S_{\downarrow}(k+2, l+1; k+3, l) \in H$. Now, $n-1 = l+1$, $n-1 = l+2$, or $n-1 \geq l+3$.

CASE 1.1: n-1=l+1. By Lemma 1.14, $L_N + (0,1) \in \text{int}(H)$. This implies that $L_N + (2,1)$ is a small cookie of H, so $e(k+3; l, l+1) \in H$. Then $e(k+3; l-2, l-1) \in H$. It follows that $L_N + (2,-2) = L_E$. But then $L_E \mapsto L_E + (0,1)$ is a short weakening of T. End of Case 1.1.

CASE 1.2: n-1= l+2. Either $e(k,k+1;l+1)\in H$ or $e(k,k+1;l+1)\notin H$.

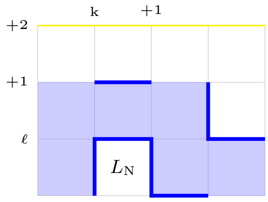

Fig. 4.35 (a). Case 1.2 (a): $L_N+(0,2)\in \text{ext}(H)$.

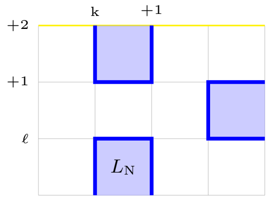

Fig. 4.35 (b). Case 1.2 (a): $L_N+(0,2)\in \text{int}(H)$.

CASE 1.2(a): $e(k,k+1;l+1)\in H$. Either $L_N+(0,2)\in \text{int}(H)$ or $L_N+(0,2)\in \text{ext}(H)$. If $L_N+(0,2)\in \text{int}(H)$, then $L_N+(0,1)\mapsto L_N$ is a short weakening of T. Suppose then, that $L_N+(0,2)\in \text{ext}(H)$. This implies that $L_N+(0,2)$ is a small cookie. Then $L_N+(0,1)\mapsto L_N+(0,2)$, $L_N\mapsto L_N+(1,1)$ is a short weakening. End of Case 1.2 (a).

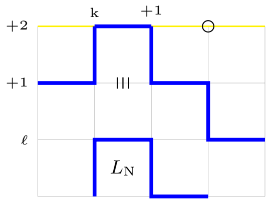

Fig. 4.36 (a). Case 1.2 (b): $e(k,k+1;l+2)\in H$.

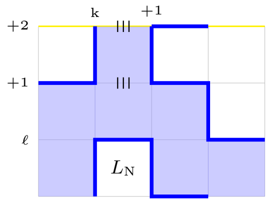

Fig. 4.36 (b). Case 1.2 (b): $e(k,k+1;l+2)\notin H$.

CASE 1.2(b): $e(k,k+1;l+1)\notin H$. Then $S_{\rightarrow}(k-1,l+1;k,l+2)\in H$ and $S_{\downarrow}(k+1,l+2;k+2,l+1)\in H$. Note that if $e(k,k+1;l+2)\in H$, then H misses $v(k+2,l+2)$, so we may assume that $e(k,k+1;l+2)\notin H$. It follows that $e(k+1,k+2;l+2)\in H$. Then $L_N+(0,2)\mapsto L_N+(1,2)$, $L_N+(0,1)\mapsto L_N$ is a short weakening. End of Case 1.2(b). End of Case 1.2.

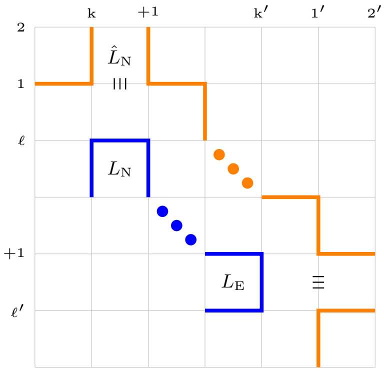

Fig. 4.37. Case 1.3(a).

CASE 1.3: $n-1\geq l+3$. By Case 1.2, we may assume that $e(k,k+1;l+1)\notin H$, $S_{\rightarrow}(k-1,l+1;k,l+2)\in H$ and that $S_{\downarrow}(k+1,l+2;k+2,l+1)\in H$. Now L_E is either open or closed.

CASE 1.3(a): L_E is closed. By previous cases and symmetry we may assume that $m-1\geq k'+3$. Using symmetry once more, we may assume that $e(k'+1;l',l'+1)\notin H$. Then the turn $\hat{T}$ on $\{e(k;l+1,l+2), S_{\downarrow}(k+1,l+2;k'+2,l'+1), e(k'+1,k'+2;l')\}$ is in H and it is a lengthening of T. End of proof of I for Case 1.3(a).

Proof of II for Case 1.3 (a). WLOG assume that the last move μ_s of a weakening $\mu_1,...,\mu_s$ of $\hat{T}$ is $Z\mapsto \hat{L}_N$, where $\hat{L}_N$ is the northern leaf of $\hat{T}$. Then $\mu_1,...,\mu_s, L_N+(0,1)\mapsto L_N$, is a weakening of T. It remains to check that $s+1\leq \min(m,n)$. Since the j^{th} lengthening T_j in $\mathcal{T}(T)$ is j units north and east of T, and $d(T)\geq 3$, there can be at most $\min(m,n)-3$ such lengthenings. Since a turn with no lengthening has a short weakening, $s+1\leq 3+\min(m,n)-3=\min(m,n)$. End of II for Case 1.3(a). End of Case 1.3(a).

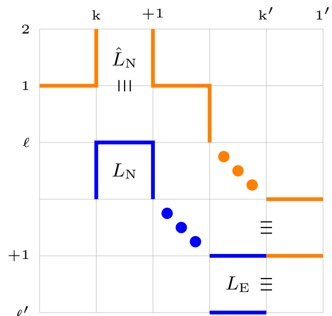

Fig. 4.38. Case 1.3(b).

CASE 1.3(b): L_E is open. Either $m-1<k'+2$ or $m-1\geq k'+2$. It will follow from Case 2 that if a turn has on open leaf adjacent to the boundary or at distance one away from the boundary, then we can find a weakening outright. Therefore, we may assume that $m-1\geq k'+2$.

If $e(k';l'+1,l'+2)\in H$, then there is a weakening $L_E\mapsto L_E+(0,1)$, so we may assume that $e(k';l'+1,l'+2)\notin H$. Then the turn $\hat{T}$ on $\{e(k;l+1,l+2), S_{\downarrow}(k+1,l+2;k'+1,l'+2), e(k',k'+1;l'+1)\}$ is in H and it is a lengthening of T. End of proof of I for Case 1.3.

*Proof of II for Case 1.3(b).*Let $\hat{L_N}$ and $\hat{L_E}$ be the northern and eastern leaves of $\hat{T}$ respectively, and let $\mu_1,...,\mu_s$ be a weakening of $\hat{T}$. If μ_s is the move $X\mapsto \hat{L}_N$, then, as in the Case 1.3(a), $\mu_1,...,\mu_s, L_N+(0,1)\mapsto L_N$, is a weakening of T. Suppose then that μ_s is the move $Z'\mapsto \hat{L}_E$. Then $\mu_1,...,\mu_s, L_E\mapsto L_E+(0,1)$, is a weakening of T. The argument that $s+1\leq \min(m,n)$ is the same as the one in Case 1.3(a), so we omit it. End of proof of II for Case 1.3(b) End of Case 1.3(b). End of Case 1.3. End of Case 1.

CASE 2: L_N is open. Proof of I. If $n-1=l$ then we must have $e(k+1,k+2;l)\in H$. Then $L_N\mapsto L_N+(1,0)$ is a weakening. Therefore, we may assume that $n-1>l$.

CASE 2.1: $n-1=l+1$. Either $e(k+1,k+2;l)\in H$ or $e(k+1,k+2;l)\notin H$.

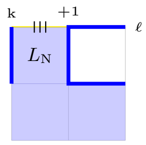

Fig. 4.39 (a). Case 2. $n-1=l$.

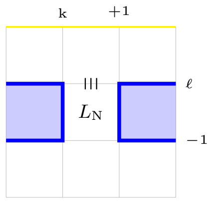

Fig. 4.39 (b). Case 2.1(a_1). $L_N+(-1,0)$ is a small cookie.

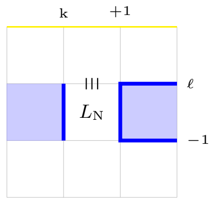

Fig. 4.39 (c). Case 2.1(a_1). $L_N+(-1,0)$ is not a small cookie.

CASE 2.1(a): $e(k+1,k+2;l)\in H$. Then $e(k,k+1;l+1)\in H$ and $e(k+1,k+2;l+1)\in H$. Then by Lemma 1.14, $L_N+(0,1)\in \text{int}H$, and so $L_N\in\text{int}H$ and $L_N+(1,0)\in\text{ext}H$. Either $k=0$, or $k>0$.

CASE 2.1(a_1): $k>0$. Then $L_N+(-1,0)\in \text{ext}H$, and $L_N+(-1,0)$ is either is a small cookie of H or it is not. If the former, then $L_N\mapsto L_N+(-1,0)$ is a short weakening; and if the latter then $L_N\mapsto L_N+(1,0)$ is a short weakening. See figures 4.39 (b) and (c). End of Case 2.1(a_1).

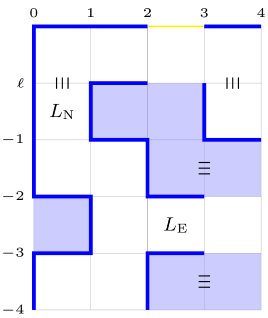

Fig. 4.40. Case 2.1(a_2)

CASE 2.1(a_2): $k=0$. Then $e(0;l,l+1)\in H$, $e(0;l-2,l-1)\in H$, and $S_{\to}(0,l-2;1,l-3)\in H$. This implies that $0\le l-4$ and that $S_{\leftarrow}(1,l-3;0,l-4)\in H$. Then we must have $S_{\uparrow}(2,l-4;3,l-3)\in H$ as well, that $L_E=L_N+(2,-2)$ and that $L_E+(0,-1)\in\text{ext}H$. See Figure 4.40. Note that if $L_E+(0,-1)$ is a small cookie, then $L_E\mapsto L_E+(0,-1)$ is a short weakening, so we may assume that $L_E+(0,-1)$ is not a small cookie. Note that this implies that $0\le l-5$.

If $e(3;l-4,l-3)\in H$, then again $L_E\mapsto L_E+(0,-1)$ is a short weakening. Similarly, if $e(3;l-2,l-1)\in H$, then $L_E\mapsto L_E+(0,1)$ is a short weakening. Therefore we only need to check the case where $e(3;l-4,l-3)\notin H$ and $e(3;l-2,l-1)\notin H$. Then $L_E+(1,-1)\in\text{ext}H$, and by the assumption that H is Hamiltonian, $m-1\ge 4$. Then we have $S_{\downarrow}(3,l;4,l-1)\in H$.

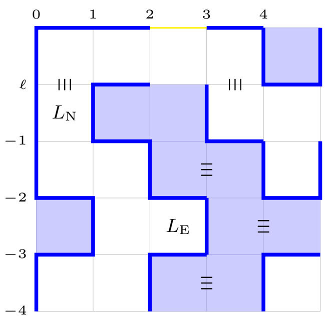

Fig. 4.41. Case 2.1(a_2).(i)

Note that either $e(2,3;l+1)\in H$ and $e(2,3;l)\in H$, or $e(2;l,l+1)\in H$ and $e(3;l,l+1)\in H$. Either way, we must have $e(3,4;l)\notin H$ and $e(3,4;l+1)\in H$. Now, either $e(3;l-3,l-2)\in H$ or $e(3;l-3,l-2)\notin H$.

CASE 2.1(a_2).(i): $e(3;l-3,l-2)\in H$. Then $L_E+(1,0)\in\text{ext}H$. By Lemma 1.14, this implies that $m-1\ge 5$. If $e(4;l-3,l-2)\in H$, then $L_E+(1,0)\mapsto L_E$ is a short weakening, so we may assume that $e(4;l-3,l-2)\notin H$. Then $S_{\downarrow}(4,l-1;5,l-2)\in H$ and $S_{\uparrow}(4,l-4;5,l-3)\in H$. We must also have that $S_{\downarrow}(4,l+1;5,l)\in H$ and $e(5;l,l+1)\in H$. Then $e(5;l-2,l-1)\in H$ as well. See Figure 4.41. Then $L_E+(2,0)\mapsto L_E+(2,1)$, $L_E+(1,0)\mapsto L_E$ is a short weakening of T. End of Case 2.1(a_2).(i).

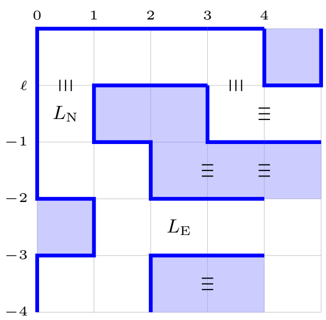

Fig. 4.42(a). Case 2.2.a_2(ii)$_1$.

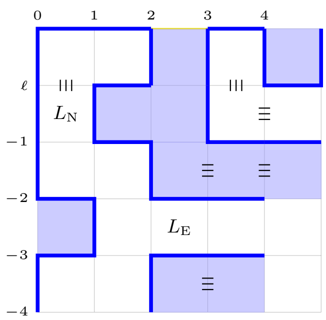

Fig. 4.42(b). Case 2.2.a_2(ii)$_2$.

CASE 2.1(a_2).(ii): $e(3;l-3,l-2)\notin H$. Then $e(3,4;l-3)\in H$ and $e(3,4;l-2)\in H$. Now, either $e(2,3;l+1)\in H$ and $e(2,3;l)\in H$, or $e(2;l,l+1)\in H$ and $e(3;l,l+1)\in H$.

CASE 2.1(a_2).(ii)$_1$: $e(2,3;l+1)\in H$ and $e(2,3;l)\in H$. Then $e(4;l-2,l-1)\notin H$. Note that if $e(4;l-1,l)\in H$, then $L_E+(1,1)\mapsto L_E+(1,2)$, $L_E\mapsto L_E+(0,1)$ is a short weakening of T, so we may assume that. $e(4;l-1,l)\notin H$. Then $e(4,5;l-1)\in H$, $S_{\downarrow}(4,l+1;5,l)\in H$, and $e(5;l,l+1)\in H$. Then $L_E+(2,2)\mapsto L_E+(2,3)$, $L_E+(1,1)\mapsto L_E+(1,2)$, $L_E\mapsto L_E+(0,1)$ is a short weakening of T. End of Case 2.1(a_2).(ii)$_1$.

CASE 2.1(a_2).(ii)$_2$: $e(2;l,l+1)\in H$ and $e(3;l,l+1)\in H$. Now, if $e(4;l-2,l-1)\notin H$, than we can use the same argument and find the same cascades as in Case 2.1(a_2).(ii)$_1$ (Fig, 4.22(b); and if $e(4;l-2,l-1)\in H$, then we must have that $e(5;l-2,l-1)\in H$ as well. Then $L_E+(2,1)\mapsto L_E+(1,1)$, Then $L_E\mapsto L_E+(0,1)$ is a short weakening of T. End of Case 2.1(a_2).(ii)$_2$. End of Case 2.1(a_2).(ii). End of Case 2.1(a_2). End of Case 2.1(a).

CASE 2.1(b): $e(k+1,k+2;l)\notin H$. Then we must have $e(k+1;l,l+1)\in H$ and $S_{\downarrow}(k+2,l+1;k+3,l)\in H$. Now, either $e(k+1,k+2;l+1)\notin H$ or $e(k+1,k+2;l+1)\in H$.

CASE 2.1(b_1): $e(k+1, k+2; l+1) \notin H$. Then we must have $e(k+2, k+3; l+1) \in H$ and that $L_N + (1,1) \in \text{ext}H$ is the neck of the large cookie. Then $k > 0$, or $k = 0$.

If $k > 0$, then after $L_N + (1,1) \mapsto L_N + (2,1)$, we are back to Case 2.1(a_1). And if $k = 0$ then we are effectively in the same scenario as in Case 2.1(a_2), with the additional, inconsequential assumption that $L_N + (1,1) \in \text{ext}H$ is the neck of the large cookie. End of Case 2.1(b_1).

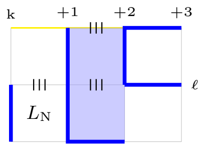

Fig. 4.43(a). Case 2.1(b_1).

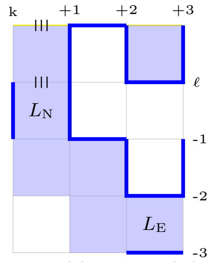

Fig. 4.43(b). Case 2.1(b_2).

CASE 2.1(b_2): $e(k+1, k+2; l+1) \in H$. Then $e(k, k+1; l+1) \notin H$ and $S_{\downarrow}(k+2, l+1; k+3, l) \in H$. Then we must have that $e(k+3; l, l+1) \in H$ as well. This implies that $e(k+3; l-2, l-1) \in H$. It follows that $L_N + (2, -2) = L_E$, and that L_E is open. Then $L_E \mapsto L_E + (0,1)$ is a short weakening. End of Case 2.1(b_2). End of Case 2.1(b) End of Case 2.1.

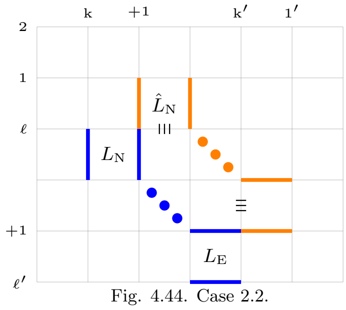

Fig. 4.44. Case 2.2.

CASE 2.2: $n-1 \geq l+2$. By previous cases we may assume that $m-1 \geq k'+2$, $e(k+1, k+2; l) \notin H$, $e(k+1; l, l+1) \in H$ and $S_{\downarrow}(k+2, l+1; k+3, l) \in H$. If L_{E} is closed, then we're done by Case 1, so we may assume that L_{E} is open. By Case 2.1, we may assume that $e(k'; l'+1, l'+2) \notin H$. Then the turn $\hat{T}$ on $\{e(k+1; l, l+1), S_{\downarrow}(k+2, l+1; k'+1, l'+2), e(k', k'+1; l'+1)\}$ is in H and it is a lengthening of T. End of proof of I for Case 2.2.

The proof of II for Case 2.2 is the same as the proof of II for Case 1.3 (a). End of proof for Case 2. □

Observation 4.12. All turn weakenings found in Lemma 4.10 are contained in Sector(T). □

Lemma 4.13. Let G be an $m \times n$ grid graph, and let H be a Hamiltonian cycle of G. Let F be a looping fat path of G, anchored at some outermost small cookie C. Then F has an admissible turn T such that Sector(T) and the j-stack of A_0's following C are disjoint.

Proof. For definiteness, assume that C is a small northern cookie, with $L = R(k'-1, l'+1)$. Let $F = G\langle N[P(X,Y)]\rangle$ be the looping H path following L, as in Lemma 4.7, with $X = R(k', l'-1)$. Let $\overrightarrow{K}$ and $e_W, e_1, ... e_s$ be as in Lemma 4.7 as well. By Lemma 4.7, F has a turn T_1. By proof of Lemma 4.7, either T_1 is a northeastern turn with X as its northern leaf, or it is not.

CASE 1: T_1 is a northeastern turn with X as its northern leaf. Then e_1 is southern. By proof of Lemma 4.8, T_2 is either southeastern or southwestern. In either case, we note that Sector(T_2) is south of the stack of A_0's. By Lemma 4.8, T_2 is admissible. End of Case 1.

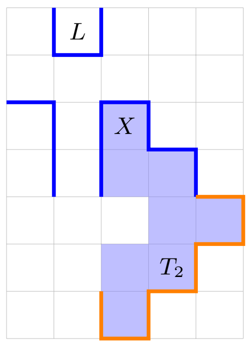

Fig. 4.45(a). An illustration of Case 1 with T_2 southeastern.

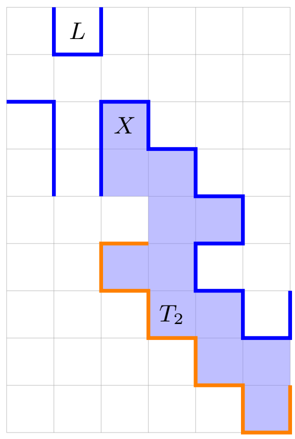

Fig. 4.45(b). An illustration of Case 1 with T_2 southwestern.

CASE 2: T_1 is not a northeastern turn with X as its northern leaf. By proof of Lemma 4.7, T_1 is a northeastern turn or T_1 is a southeastern turn.

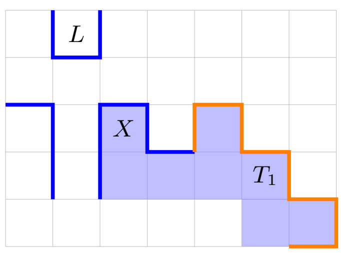

Fig. 4.46(a). An illustration of Case 2.1.

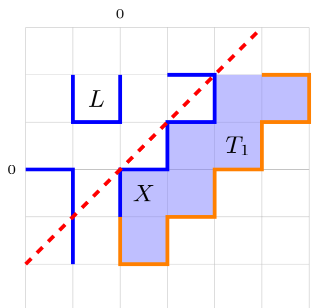

Fig. 4.46(b). An illustration of Case 2.2. The line $x = y$ in red.

CASE 2.1: T_1 is a northeastern turn. It follows from the proof of Lemma 4.7 that Sector(T_1) is east of the stack of A_0's, that both leaves of T_1 are in F, and that neither leaf of T_1 is an end-box of $P(X,Y)$. End of Case 2.1.

CASE 2.2: T_1 is a southeastern turn. It follows from the proof of Lemma 4.7 that Sector(T_1) is southeast of the stack of A_0's. More precisely, we can check that Sector(T_1) is below the line $y = x + (l' - k')$, and the j-stack of A_0's is above the line $y = x + (l' - k')$. By proof of Lemma 4.7, we have that both leaves of T_1 are in F, and that no leaf of T_1 is an end-box of $P(X, Y)$. End of Case 2.2. End of Case 2. □

Now we are ready to give a proof of Lemma 3.13.

Lemma 3.13. Let G be an $m \times n$ grid graph, and let H be a Hamiltonian cycle of G. Let C be a small cookie of G. Assume that G has only one large cookie, and that there is a j-stack of A_0 starting at the A_0-type containing C. Let L be the leaf in the top (j^{th}) A_0 of the stack, and assume that L is followed by an A_1-type. Let X and Y be the boxes adjacent to the middle-box of the A_1-type that are not its H-neighbours. If $P(X, Y)$ has no switchable boxes, then either:

(i) there is a cascade of length at most $\min(m, n)$, which avoids the stack of A_0's, and after which $P(X, Y)$ gains a switchable box, or

(ii) there is a cascade of length at most $\min(m, n) + 1$, that collects L and avoids the stack of A_0's.

Proof. Suppose that $P(X, Y)$ has no switchable boxes. Then $P(X, Y)$ is contained in a looping fat path $F = G\langle N[P(X, Y)]\rangle$. By Lemma 4.13, F has an admissible turn T such that Sector(T) and the j-stack of A_0's are disjoint. Then, by Corollary 4.9(a), $d(T) \geq 3$. By Proposition 4.11, T has a weakening $\mu_1, ..., \mu_s$. By Observation 4.12, $\mu_1, ..., \mu_s$ is contained in Sector(T), and thus it avoids the j-stack of A_0's.

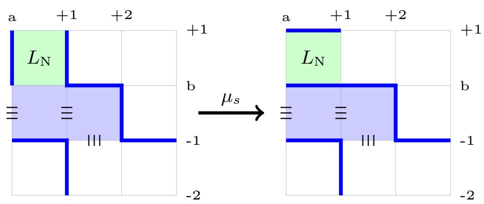

Fig. 4.47(a). $L_N+(0,-1)$ is not an end-box. Fig. 4.47(b). After μ_s.

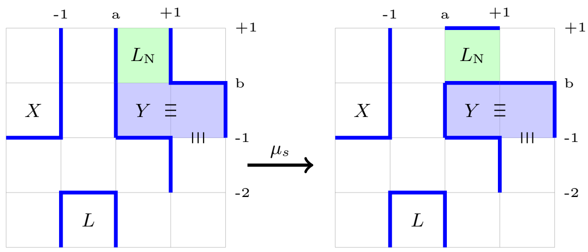

Fig. 4.48(a). $L_N+(0,-1)$ is an end-box. Fig. 4.48(b). After μ_s.

For definiteness, assume that T is northeastern with northern leaf $L_N = R(a, b)$ and that μ_s is the move $L_N \mapsto L'_N$ or the move $L'_N \mapsto L_N$. By Corollary 4.9(b), $L_N \in N[P] \setminus P$ and $L_N + (0, -1) \in P$. Note that we must have $S_{\rightarrow}(a, b-1; a+1, b-2) \in H$, $e(a, a+1; b) \notin H$ and $e(a+1; b-1, b) \notin H$. Now, either $L_N + (0, -1)$ is an end-box of P or it is not. If the latter then, $e(a; b-1, b \notin H)$. Then, after μ_s, $L_N + (0, -1) \in P(X, Y)$ is switchable. The fact that $s \leq \min(m, n)$, follows immediately from Proposition 4.11. Thus, in this case, (i) holds.

Suppose then, that $L_N + (0, -1)$ is an end-box of P, say Y. This implies that $e(a; b-1, b) \in H$, that F is northern, and that $L = L_N + (-1, -3)$. Then μ_s can be followed by $Y + (-1, 0) \mapsto Y$, $L + (0, 1) \mapsto L$, which collects L. To check the length of the cascade, first we note that $b + 1 \geq 4$, and that the eastern leaf of T has x-coordinate at least 5. It follows that $|\mathcal{T}(T)| \leq \min(m, n) - 4$. It follows from the proof of Lemma 4.10 that T has a weakening of length at most $\min(m, n) - 4 + 3$. Since we need two additional moves after μ_s to collect L, the length of the cascade is at most $\min(m, n) - 4 + 3 + 2 = \min(m, n) + 1$ moves. Thus, in this case, (ii) holds. □

Acknowledgments

This paper is based on part of the author's PhD dissertation at York University, written under the supervision of Professor Neal Madras.

The author is grateful to his advisor, Neal Madras, for his patience and guidance through several rewrites, and for his thoughtful and insightful feedback on each of them. The author also thanks Nathan Clisby for creating an interactive online tool for visualizing and experimenting with Hamiltonian paths and cycles[3].

This research was supported in part by a Discovery Grant from NSERC Canada to my advisor Neal Madras.

Declarations

Funding

The author received no external funding for this work.

Competing interests

The author has no relevant financial or non-financial interests to disclose.

Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Authors' contributions

The author is the sole contributor to all aspects of this work.

References

[1] Albi Kazazi. Reconfiguration of hamiltonian cycles in grid graphs, 2024. YouTube video.

[2] Shao Dong Chen, Hong Shen, and Rodney Topor. An efficient algorithm for constructing hamiltonian paths in meshes. *Parallel Computing*, 28(9):1293–1305, 2002.

[3] Nathan Clisby. Hamiltonian path and cycle visualizations, 2025.

[4] JM Deutsch. Long range moves for high density polymer simulations. *The Journal of Chemical Physics*, 106(21):8849–8854, 1997.

[5] Alon Itai, Christos H Papadimitriou, and Jayme Luiz Szwarcfiter. Hamilton paths in grid graphs. *SIAM Journal on Computing*, 11(4):676–686, 1982.

[6] Jesper Lykke Jacobsen. Unbiased sampling of globular lattice proteins in three dimensions. *Physical Review Letters*, 100(11):118102, 2008.

[7] Albi Kazazi. Reconfiguration of hamiltonian cycles and paths in grid graphs. phd dissertation. york university, 2025.

[8] Neal Madras and Gordon Slade. *The Self-Avoiding Walk*. Springer Science & Business Media, 2013.

[9] Marc L Mansfield. Monte carlo studies of polymer chain dimensions in the melt. *The Journal of Chemical Physics*, 77(3):1554–1559, 1982.

[10] Rahnuma Islam Nishat. *Reconfiguration of Hamiltonian cycles and paths in grid graphs*. PhD thesis, University of Victoria, 2020.

[11] Rahnuma Islam Nishat, Venkatesh Srinivasan, and Sue Whitesides. 1-complex s, t hamiltonian paths: Structure and reconfiguration in rectangular grids. *Journal of Graph Algorithms and Applications*, 27(4):281–327, 2023.

[12] Rahnuma Islam Nishat, Venkatesh Srinivasan, and Sue Whitesides. The hamiltonian path graph is connected for simple s, t paths in rectangular grid graphs. *Journal of Combinatorial Optimization*, 48(4):31, 2024.

[13] Rahnuma Islam Nishat and Sue Whitesides. Bend complexity and hamiltonian cycles in grid graphs. In *International Computing and Combinatorics Conference*, pages 445–456. Springer, 2017.

[14] Rahnuma Islam Nishat and Sue Whitesides. Reconfiguring hamiltonian cycles in l-shaped grid graphs. In *International Workshop on Graph-Theoretic Concepts in Computer Science*, pages 325–337. Springer, 2019.

[15] Rahnuma Islam Nishat and Sue Whitesides. Reconfiguration of hamiltonian cycles in rectangular grid graphs. *International Journal of Foundations of Computer Science*, 34(07):773–793, 2023.

[16] Richard Oberdorf, Allison Ferguson, Jesper L Jacobsen, and Jané Kondev. Secondary structures in long compact polymers. *Physical Review E—Statistical, Nonlinear, and Soft Matter Physics*, 74(5):051801, 2006.

[17] Christopher Umans and William Lenhart. Hamiltonian cycles in solid grid graphs. In *Proceedings 38th Annual Symposium on Foundations of Computer Science*, pages 496–505. IEEE, 1997.